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A
KEY
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NEW HIGHER ALGEBRA.

By BENJAMIN GREENLEAF, A. M.
AUTHOR OF A MATHEMATICAL SERIES.

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KEY

TO

GREENLEAF'S NEW HIGHER ALGEBRA.

DEFINITIONS AND NOTATION.

EXERCISES ON DEFINITIONS AND PRINCIPLES.

(ART. **48**, pp. 17, 18.)

2. Ans. $2x - y^2$.
4. Ans. $ab \times c^2 \times d^3$, or $ab \, . \, c^2 \, . \, d^3$, or abc^2d^3.
5. Ans. $\frac{x+y}{a-b}$.
6. Ans. $a^2 \sim b^2$.
8. Ans. $\frac{5}{2a} + a^2 - b^2 = \sqrt{c^2 + d^2}$.
9. Ans. $\frac{5a}{b} : \frac{d}{c^3} = x^2 y^3 : y^2 z^4$.
10. Ans. $\sqrt[3]{\frac{c-d}{a+b}} > (a-d)^2 (a+b)^3 - \frac{3x}{4}$.

(ART. **49**, pp. 18, 19.)

3. $72 + 60 - 5 = 127$, Ans.
4. $36 \times 11 - 240 = 396 - 240 = 156$, Ans.
5. $900 - 1500 + 108 = -492$, Ans.
6. $252 + (-1) \times 2 = 252 - 2 = 250$, Ans.

7. $60 - 35 - 1 = 24$, Ans.

8. $240 - 91 + 1 = 150$, Ans.

9. $2 + \frac{2}{3} + 3 = 5\frac{2}{3}$, Ans.

10. $4 + \frac{4}{9} + 9 = 13\frac{4}{9}$, Ans.

11. $\frac{4}{3} + \frac{8}{3} = \frac{12}{3} = 4$, Ans.

12. $\frac{100 - 90 - 1}{2 + 3} = \frac{9}{5} = 1\frac{4}{5}$, Ans.

13. $(1 + 1 + 1)(4 + 1 - 1)(1 - 1 + 3) = 3 \times 4 \times 3 = 36$, Ans.

14. $8 - 20 + 2 + 18 = 8$.

15. $16 - 16 + 3\frac{1}{3} - \frac{2}{5} + \frac{1}{15} = 3$, Ans.

16. $\frac{\frac{5}{3} + \frac{1}{3} - [\frac{3}{3} - \frac{1}{3}]}{\frac{1}{3}} = 5 + 1 - 2 = 4$.

17. $\frac{\frac{13}{3} + \frac{3}{3} - (\frac{14}{3} - \frac{3}{3})}{\frac{2}{3} + \frac{3}{3}} = \frac{13 + 3 - 11}{2 + 3} = \frac{5}{5} = 1$, Ans.

18. $9 - 2 + 2 = 9$.

19. $6 + 6 - 2 = 10$, Ans.

20. $5 \times 14 + \sqrt{6 \times 6} = 70 + 6 = 76$.

ADDITION.

(Arts. 54, 55, pp. 21–23.)

3. Ans. $8ax$.

4. Ans. $7n$.

(7.)

$$\begin{array}{l}
3a + b - 10 \\
-a + c - d \\
+2a - 3b - 7 - 4c \\
 5 + 4x^2 - 18m \\
\hline
4a - 2b - 12 - 3c - d + 4x^2 - 18m
\end{array}$$

(8.)

$$\begin{array}{rrr}
7a & -5y^3 & \\
2a & & +8\sqrt{x} \\
 & 5y^3 & -\sqrt{x} \\
-9a & & +7\sqrt{x} \\ \hline
 & & 14\sqrt{x}
\end{array}$$

(9.)

$$\begin{array}{rrrrrr}
4mn & +3ab & -4c & & & \\
2mn & -4ab & & +3x & & \\
 & & & & +3m^2 & -4p \\ \hline
6mn & -ab & -4c & +3x & +3m^2 & -4p
\end{array}$$

(10.)

$$\begin{array}{rrr}
3a^2 & +2ab & +4b^2 \\
5a^2 & -8ab & +b^2 \\
-a^2 & +5ab & -b^2 \\
18a^2 & -20ab & -19b^2 \\
14a^2 & -3ab & +20b^2 \\ \hline
39a^2 & -24ab & +5b^2
\end{array}$$

(11.)

$$\begin{array}{rrrr}
4x^3 & -5a^3 & -5ax^2 & +6a^2x \\
3x^3 & +6a^3 & +4ax^2 & +2a^2x \\
-17x^3 & & +19ax^2 & -15a^2x \\
 & 18a^3 & +13ax^2 & -27a^2x \\
12x^3 & -20a^3 & & +3a^2x \\
-9x^3 & & -31ax^2 & +31a^2x \\ \hline
-7x^3 & -a^3 & &
\end{array}$$

(12.)

$$\begin{array}{rr}
3ab & +3(a+b) \\
-ab & +2(a+b) \\
7ab & -4(a+b) \\
-2ab & +6(a+b) \\ \hline
7ab & +7(a+b)
\end{array}$$

(13.)

$$\begin{array}{lrr}
 & 7\sqrt{y} & -4(a+b) \\
 & 6\sqrt{y} & +2(a+b) \\
 & 2\sqrt{y} & +(a+b) \\
 & \sqrt{y} & -3(a+b) \\ \cline{2-3}
\text{Ans.} & 16\sqrt{y} & -4(a+b)
\end{array}$$

(14.)

$$\begin{array}{rrrrrr}
3(c+d)x^3 & +4y & -2\sqrt{y} & & & \\
 & & & 6x^2y & -2ax & +12 \\
 & y & +\sqrt{y} & & -5ax & \\
(c+d)x^3 & & & -x^2y & +3ax & \\
 & 2y & & +x^2y & & -14 \\ \hline
4(c+d)x^3 & +7y & -\sqrt{y} & +6x^2y & -4ax & -2
\end{array}$$

(17.)

$$\begin{array}{r}
amx+2dy \\
2cx-3dy \\
3dx+5y \\
\hline
(am+2c+3d)x+(5-d)y
\end{array}$$

(18.)

$$\begin{array}{r}
4ax-5my \\
3dx+7ny \\
7mx+4my \\
\hline
(4a+3d+7m)x+(7n-m)y
\end{array}$$

(19.)

$$\begin{array}{r}
3nz-5x \\
4mz+nx \\
5az-4mx \\
\hline
(3n+4m+5a)z+(n-5-4m)x
\end{array}$$

(20.)

$$\begin{array}{r}
ax+5y \\
6ax-y \\
2bx+2y \\
\hline
(7a+2b)x+6y
\end{array}$$

(21.)

$$\begin{array}{r}
(2a+c)(m-\sqrt{m}) \\
b(m-\sqrt{m}) \\
-3c(m-\sqrt{m}) \\
\hline
(2a+b-2c)(m-\sqrt{m})
\end{array}$$

(22.)

$$\begin{array}{lll}
ax+dy^2 & & \\
-dx & & +by \\
 & -by^2 & +my \\
\hline
(a-d)x & +(d-b)y^2 & +(b+m)y
\end{array}$$

SUBTRACTION.

(ARTS. **59-61,** pp. 25-28.)

(9.)

$$\begin{array}{lr}
 & 31x^2-3y^2+ab \\
 & 17x^2+5y^2-4ab+7 \\
\hline
\text{Ans.} & 14x^2-8y^2+5ab-7
\end{array}$$

(11.)

$$\begin{array}{r}
6a+3b-5c+1 \\
6a-3b-5c \\
\hline
\text{Ans. } 6b \qquad +1
\end{array}$$

(12.)

$$\begin{array}{lr}
 & 5xy+4a^2-6b^2 \\
 & 3xy-9a^2-b^2+4 \\
\hline
\text{Ans.} & 2xy+13a^2-5b^2-4
\end{array}$$

(13.)

$$\begin{array}{r}
11a-7b+c \\
a+7b-3c+11 \\
\hline
10a-14b+4c-11
\end{array}$$

(14.)

$$\begin{array}{r} m^2+3n^3 \\ -4m^2-6n^3+71x \\ \hline 5m^2+9n^3-71x \end{array}$$

(15.)

$$\begin{array}{r} 31a-15x-7 \\ 2a-25x \quad +y^2 \\ \hline 29a+10x-7-y^2 \end{array}$$

(16.)

$$\begin{array}{r} a+\ b \\ 2a-2b \\ \hline -a+3b \\ -a+\ b \\ \hline 2b \end{array}$$

(17.)

$$\begin{array}{r} a-\ b-\ c \\ -a+\ b+\ c \\ \hline 2a-2b-2c \\ a-\ b+\ c \\ \hline a-\ b-3c \end{array}$$

19. Ans. $a-(b-c-d-e)$.

20. $3a+b-a+b=2a+2b$.

21. $2a^3-3a^2b+4ab^2-a^3-b^3-ab^2$
$=a^3-3a^2b+3ab^2-b^3$.

22. $6a-9x-4a+5x=2a-4x$, Ans.

23. $4a-5x-a+4x+x-8a=-5a$.

25. $5a-4b+3c-3a+2b-c=2a-2b+2c$.

26. $6x^2+2y^2-3x^2-y^2-2x^2-4y^2+4x^2-y^2$
$=5x^2-4y^2$.

27. $1-2+7-x+4+2-3-4+x-5=0$.

(28.)

$$\begin{array}{r} ay-b \\ dy \quad -c \\ \hline (a-d)y-b+c \end{array}$$

(29.)

$$\begin{array}{r} ax^3+dy^2 \\ bx^3-cy^2 \\ \hline (a-b)x^3+(d+c)y^2 \end{array}$$

(30.)

$$\begin{array}{r} ax-bx+cx \\ ax-bx+\ x \\ \hline (c-1)x \end{array}$$

(31.)

$$\begin{array}{r} ax^3-\ bx^2+cx-d \\ bx^2+ex-2d \\ \hline ax^3-2bx^2+(c-e)x+d \end{array}$$

(32.)

$$\begin{array}{rrr}
(4a - 5b)\sqrt{x+y} & +\ (a - \ \ b)\,xy & -\qquad c\ z^2 \\
-\ (3a - 2b)\sqrt{x+y} & +\qquad 3b\ \,xy & -\ (5 + c)\,z^2 \\
\hline
(7a - 7b)\sqrt{x+y} & +\ (a - 4b)\,xy & +\qquad 5\,z^2
\end{array}$$

MULTIPLICATION.

CASE I.

(Art. **67**, pp. 30, 31.)

5. Ans. $45\,m^6\,n^7$. 11. Ans. $-55\,n^8$.

17. $8\,a\,x^2\,y^2 \times (-2\,a^3\,x^4\,y^3) = -16\,a^4\,x^6\,y^5$;
$-16\,a^4\,x^6\,y^5 \times (-4\,a^2\,x\,y^3) = 64\,a^6\,x^7\,y^8$.

19. $-2\,a\,c^2 \times (-4\,a\,c^{-3}) = 8\,a^2\,c^{-1}$; $8\,a^2\,c^{-1} \times (-3\,a\,b^2)$
$= -24\,a^3\,b^2\,c^{-1}$; $-24\,a^3\,b^2\,c^{-1} \times (-b\,c\,d) = 24\,a^3\,b^3\,d$.

CASE II.

(Arts. **68**, **69**, pp. 31–33.)

(8.)

$$\begin{array}{llll}
3x^2 & -\ \ 2xy & -\ y^2 & \\
2x & -\ \ 4y & & \\
\hline
6x^3 & -\ \ 4x^2y & -\ 2xy^2 & \\
 & -\ 12x^2y & +\ 8xy^2 & +\ 4y^3 \\
\hline
6x^3 & -\ 16x^2y & +\ 6xy^2 & +\ 4y^3
\end{array}$$

(9.)

$$\begin{array}{lllll}
x^2 & +\ 2x & +\ 1 & & \\
x^2 & -\ 2x & +\ 3 & & \\
\hline
x^4 & +\ 2x^3 & +\ x^2 & & \\
 & -\ 2x^3 & -\ 4x^2 & -\ 2x & \\
 & & 3x^2 & +\ 6x & +\ 3 \\
\hline
x^4 & & & +\ 4x & +\ 3
\end{array}$$

(10.)

$$\begin{array}{llllll}
a & +b & -c & & & \\
a & -b & +c & & & \\ \hline
a^2 & +ab & -ac & & & \\
 & -ab & & -b^2 & +bc & \\
 & & ac & & +bc & -c^2 \\ \hline
a^2 & & & -b^2 & +2bc & -c^2
\end{array}$$

(11.)

$$\begin{array}{lrrr}
 & 3a & -2b & \\
 & -2a & +4b & \\ \hline
 & -6a^2 & +4ab & \\
 & & 12ab & -8b^2 \\ \hline
\text{Ans.} & -6a^2 & +16ab & -8b^2
\end{array}$$

(12.)

$$\begin{array}{llll}
a^2 & +ab & +b^2 & \\
a & -b & & \\ \hline
a^3 & +a^2b & +ab^2 & \\
 & -a^2b & -ab^2 & -b^3 \\ \hline
a^3 & & & -b^3
\end{array}$$

(13.)

$$\begin{array}{llllll}
 & 1 & -x & +x^2 & -x^3 & \\
 & a & +ax & & & \\ \hline
 & a & -ax & +ax^2 & -ax^3 & \\
 & & ax & -ax^2 & +ax^3 & -ax^4 \\ \hline
\text{Ans.} & a & & & & -ax^4
\end{array}$$

(14.)

$$\begin{array}{llll}
5a^2 & -3ab & +4b^2 & \\
6a & -5b & & \\ \hline
30a^3 & -18a^2b & +24ab^2 & \\
 & -25a^2b & +15ab^2 & -20b^3 \\ \hline
30a^3 & -43a^2b & +39ab^2 & -20b^3
\end{array}$$

(15.)

$$\begin{array}{llll}
3a & -x & & \\
2a & +4x & & \\ \hline
6a^2 & -2ax & & \\
 & 12ax & -4x^2 & \\ \hline
6a^2 & +10ax & -4x^2 & \\
4a & -2x & & \\ \hline
24a^3 & +40a^2x & -16ax^2 & \\
 & -12a^2x & -20ax^2 & +8x^3 \\ \hline
24a^3 & +28a^2x & -36ax^2 & +8x^3
\end{array}$$

(17.)

$$\begin{array}{lll}
3a & +4b & \\
2a & +b & \\ \hline
6a^2 & +8ab & \\
 & 3ab & +4b^2 \\ \hline
6a^2 & +11ab & +4b^2
\end{array}$$

(18.)

$$\begin{array}{llllll}
a^4 & -a^3x & +a^2x^2 & -ax^3 & +x^4 & \\
a & +x & & & & \\ \hline
a^5 & -a^4x & +a^3x^2 & -a^2x^3 & +ax^4 & \\
 & a^4x & -a^3x^2 & +a^2x^3 & -ax^4 & +x^5 \\ \hline
a^5 & & & & & +x^5
\end{array}$$

(19.)

$$\begin{array}{l}
a^4 - x^4 \\
a^4 - x^4 \\
\hline
a^8 - a^4x^4 \\
\quad - a^4x^4 + x^8 \\
\hline
a^8 - 2a^4x^4 + x^8
\end{array}$$

(20.)

$$\begin{array}{l}
a^m - a^n \\
2a - a^n \\
\hline
2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}
\end{array}$$

(21.)

$$\begin{array}{l}
1 - x + x^2 - x^3 \\
1 + x \\
\hline
1 - x + x^2 - x^3 \\
\quad\quad x - x^2 + x^3 - x^4 \\
\hline
1 - x^4 \\
1 + x^4 \\
\hline
1 - x^4 \\
\quad\quad x^4 - x^8 \\
\hline
1 \quad\quad - x^8
\end{array}$$

(22.)

$$\begin{array}{l}
a + 2x \\
a - 3x \\
\hline
a^2 + 2ax \\
\quad - 3ax - 6x^2 \\
\hline
a^2 - ax - 6x^2 \\
a + 4x \\
\hline
a^3 - a^2x - 6ax^2 \\
\quad\quad 4a^2x - 4ax^2 - 24x^3 \\
\hline
a^3 + 3a^2x - 10ax^2 - 24x^3
\end{array}$$

(23.)

$$\begin{array}{l}
a^3 - 3a^2 + 3a - 1 \\
a^2 - 2a + 1 \\
\hline
a^5 - 3a^4 + 3a^3 - a^2 \\
\quad - 2a^4 + 6a^3 - 6a^2 + 2a \\
\quad\quad\quad\quad a^3 - 3a^2 + 3a - 1 \\
\hline
a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1
\end{array}$$

MULTIPLICATION BY DETACHED COEFFICIENTS.

(Art. **74**, pp. 35, 36.)

(3.)

$$\begin{array}{l}
1 + 1 - 3 \\
1 + 0 - 1 \\
\hline
1 + 1 - 3 \\
\quad\quad - 1 \quad - 1 + 3 \\
\hline
1 + 1 - 4 \quad - 1 + 3 \\
y^5 + y^4 - 4y^3 - y^2 + 3y
\end{array}$$

(4.)

$$\begin{array}{l}
1 + 1 + 1 + 1 + 1 + 1 \\
1 - 1 \\
\hline
1 + 1 + 1 + 1 + 1 + 1 \\
\quad - 1 - 1 - 1 - 1 - 1 - 1 \\
\hline
1 + 0 + 0 + 0 + 0 + 0 - 1 \\
\quad\quad\quad x^6 - 1
\end{array}$$

(5.)

$$\begin{array}{rrrrr}
1 & -2 & +4 & & \\
1 & +2 & +4 & & \\ \hline
1 & -2 & +4 & & \\
 & 2 & -4 & +8 & \\
 & & 4 & -8 & +16 \\ \hline
1 & +0 & +4 & +0 & +16
\end{array}$$

$$a^4+4a^2b^2+16b^4$$

(6.)

$$\begin{array}{rrrrrr}
3 & +3 & +3 & +3 & +3 & \\
7 & -7 & & & & \\ \hline
21 & +21 & +21 & +21 & +21 & \\
 & -21 & -21 & -21 & -21 & -21 \\ \hline
21 & +0 & +0 & +0 & +0 & -21
\end{array}$$

$$21a^5-21b^5$$

(7.)

$$\begin{array}{rrrrr}
1 & +1 & +1 & +1 & \\
1 & -1 & & & \\ \hline
1 & +1 & +1 & +1 & \\
 & -1 & -1 & -1 & -1 \\ \hline
1 & +0 & +0 & +0 & -1
\end{array}$$

$$x^4-y^4$$

DIVISION.

CASE I.

(Art. **79**, pp. 39, 40.)

5. Ans. $3y^2$.

8. $3a^4c-a+6a^2c$.

15. $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=\frac{a^{\frac{3}{6}}}{a^{\frac{2}{6}}}=a^{\frac{1}{6}}$.

CASE II.

(Art. **80**, pp. 40–43.)

(4.)

$$\begin{array}{rrrr|l}
8a^3 & -4a^2b & -6ab^2 & +3b^3 & 2a-b \\ \cline{5-5}
8a^3 & -4a^2b & & & 4a^2-3b^2 \\ \cline{1-4}
 & & -6ab^2 & +3b^3 & \\
 & & -6ab^2 & +3b^3 & \\ \cline{3-4}
\end{array}$$

(5.)

$$\begin{array}{r|l}
-4a^3-4a^2b+3ab^2+3b^3 & \underline{a+b} \\
\underline{-4a^3-4a^2b\qquad\qquad\qquad} & -4a^2+3b^2 \\
3ab^2+3b^3 & \\
\underline{3ab^2+3b^3} &
\end{array}$$

(6.)

$$\begin{array}{r|l}
2a^2x^2-5ax+2 & \underline{2ax-1} \\
\underline{2a^2x^2-ax} & ax-2 \\
-4ax+2 & \\
\underline{-4ax+2} &
\end{array}$$

(7.)

$$\begin{array}{l|l}
21a^5-21b^5 & \underline{7a-7b} \\
\underline{21a^5-21a^4b} & 3a^4+3a^3b+3a^2b^2+3ab^3+3b^4
\end{array}$$

$$\begin{array}{r}
21a^4b-21b^5\qquad\qquad\qquad\qquad\qquad\qquad \\
\underline{21a^4b-21a^3b^2}\qquad\qquad\qquad\qquad\qquad\qquad \\
21a^3b^2-21b^5\qquad\qquad\qquad\qquad \\
\underline{21a^3b^2-21a^2b^3}\qquad\qquad\qquad\qquad \\
21a^2b^3-21b^5\qquad\qquad \\
\underline{21a^2b^3-21ab^4}\qquad\qquad \\
21ab^4-21b^5 \\
\underline{21ab^4-21b^5}
\end{array}$$

(8.)

$$\begin{array}{r}
x^2+y^2-z^2)\,x^4-y^4+2y^2z^2-z^4\,(x^2-y^2+z^2 \\
\underline{x^4+x^2y^2-x^2z^2}\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
-x^2y^2-y^4+x^2z^2+2y^2z^2\qquad\qquad\qquad \\
\underline{-x^2y^2-y^4\qquad\qquad+y^2z^2}\qquad\qquad\qquad \\
x^2z^2+y^2z^2-z^4\qquad \\
\underline{x^2z^2+y^2z^2-z^4}\qquad
\end{array}$$

(9.)

$$\begin{array}{l|l}
x^4+y^4 & x+y \\ \cline{2-2}
\underline{x^4+x^3y} & x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y} \\
\quad -x^3y+y^4 & \\
\quad \underline{-x^3y-x^2y^2} & \\
\qquad\qquad x^2y^2+y^4 & \\
\qquad\qquad \underline{x^2y^2+xy^3} & \\
\qquad\qquad\qquad -xy^3+y^4 & \\
\qquad\qquad\qquad \underline{-xy^3-y^4} & \\
\qquad\qquad\qquad\qquad 2y^4 &
\end{array}$$

(10.)

$$\begin{array}{l|l}
a^3+5a+5a^{-1}+a^{-3} & a+a^{-1} \\ \cline{2-2}
\underline{a^3+\ \ a} & a^2+4+a^{-2} \\
\qquad 4a+5a^{-1} & \\
\qquad \underline{4a+4a^{-1}} & \\
\qquad\qquad a^{-1}+a^{-3} & \\
\qquad\qquad \underline{a^{-1}+a^{-3}} &
\end{array}$$

(11.)

$$\begin{array}{l}
a+x)\ a^3-x^3\ (\,a^2-ax+x^2-\dfrac{2x^3}{a+x} \\
\qquad\quad \underline{a^3+a^2x} \\
\qquad\qquad -a^2x-x^3 \\
\qquad\qquad \underline{-a^2x-ax^2} \\
\qquad\qquad\qquad\quad ax^2-x^3 \\
\qquad\qquad\qquad\quad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\qquad -2x^3
\end{array}$$

(12.)

$$\begin{array}{l|l}
x^{n+1}+x^ny+xy^n+y^{n+1} & x^n+y^n \\ \cline{2-2}
x^{n+1}\qquad\quad +xy^n & x+y \\ \cline{1-1}
\qquad x^ny\qquad\quad +y^{n+1} & \\
\qquad \underline{x^ny\qquad\quad +y^{n+1}} &
\end{array}$$

(13.)

$$\begin{array}{l}
x^2+3x+y\,)\,x^4-9x^2-6xy-y^2\,(\,x^2-3x-y \\
\qquad\qquad\quad \underline{x^4+3x^3+x^2y} \\
\qquad\qquad\qquad \underline{-3x^3-9x^2-x^2y-6xy} \\
\qquad\qquad\qquad \underline{-3x^3-9x^2\qquad\quad -3xy} \\
\qquad\qquad\qquad\qquad\qquad\quad -x^2y-3xy-y^2 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-x^2y-3xy-y^2}
\end{array}$$

(14.)

$$\begin{array}{l}
1-a\,)\,1+a\,(\,1+2a+2a^2+2a^3+\text{ etc.} \\
\qquad\quad \underline{1-a} \\
\qquad\qquad 2a \\
\qquad\qquad \underline{2a-2a^2} \\
\qquad\qquad\qquad 2a^2 \\
\qquad\qquad\qquad \underline{2a^2-2a^3} \\
\qquad\qquad\qquad\qquad 2a^3 \\
\qquad\qquad\qquad\qquad \underline{2a^3-2a^4} \\
\qquad\qquad\qquad\qquad\qquad 2a^4
\end{array}$$

(15.)

$$\begin{array}{l}
1+x\,)\,a\qquad\quad (\,a-ax+ax^2-ax^3+\text{ etc.} \\
\qquad\quad \underline{a+ax} \\
\qquad\qquad -ax \\
\qquad\qquad \underline{-ax-ax^2} \\
\qquad\qquad\qquad ax^2 \\
\qquad\qquad\qquad \underline{ax^2+ax^3} \\
\qquad\qquad\qquad\quad -ax^3 \\
\qquad\qquad\qquad\quad \underline{-ax^3-ax^4} \\
\qquad\qquad\qquad\qquad\quad ax^4
\end{array}$$

(16.)

$$\begin{array}{l|l}
a^{m+n}+a^{n+1}b^{m-1}-a^{m-1}b^{n+1}-b^{m+n} & \underline{a^{n+1}-b^{n+1}} \\
\underline{a^{m+n}\qquad\qquad\qquad -a^{m-1}b^{n+1}\qquad\quad} & a^{m-1}+b^{m-1} \\
\qquad a^{n+1}b^{m-1}\qquad\qquad\qquad -b^{m+n} & \\
\qquad \underline{a^{n+1}b^{m-1}\qquad\qquad\qquad -b^{m+n}} &
\end{array}$$

(17.)

$$\begin{array}{l|l}
a^8+a^6b^2+a^4b^4+a^2b^6+b^8 & a^4+a^3b+a^2b^2+ab^3+b^4 \\ \cline{2-2}
\underline{a^8+a^7b+a^6b^2+a^5b^3+a^4b^4} & a^4-a^3b+a^2b^2-ab^3+b^4 \\
\multicolumn{1}{l}{\quad -a^7b-a^5b^3+a^2b^6+b^8} & \\
\multicolumn{1}{l}{\quad \underline{-a^7b-a^6b^2-a^5b^3-a^4b^4-a^3b^5}} & \\
\multicolumn{1}{l}{\qquad\qquad a^6b^2+a^4b^4+a^3b^5+a^2b^6+b^8} & \\
\multicolumn{1}{l}{\qquad\qquad \underline{a^6b^2+a^5b^3+a^4b^4+a^3b^5+a^2b^6}} & \\
\multicolumn{1}{l}{\qquad\qquad\quad -a^5b^3+b^8} & \\
\multicolumn{1}{l}{\qquad\qquad\quad \underline{-a^5b^3-a^4b^4-a^3b^5-a^2b^6-ab^7}} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad a^4b^4+a^3b^5+a^2b^6+ab^7+b^8} & \\
\multicolumn{1}{l}{\qquad\qquad\qquad\qquad \underline{a^4b^4+a^3b^5+a^2b^6+ab^7+b^8}} &
\end{array}$$

DIVISION BY DETACHED COEFFICIENTS.

(Art. 81, pp. 43–45.)

(3.)

$$\begin{array}{l|l}
6-17+0+16 & 3-4 \\ \cline{2-2}
\underline{6-8} & 2-3-4 \\
\multicolumn{1}{l}{\quad -9+0} & 2x^2-3xy-4y^2 \\
\multicolumn{1}{l}{\quad \underline{-9+12}} & \\
\multicolumn{1}{l}{\qquad\quad -12+16} & \\
\multicolumn{1}{l}{\qquad\quad \underline{-12+16}} &
\end{array}$$

(4.)

$$\begin{array}{l|l}
3+3-4-4 & 1+1 \\ \cline{2-2}
\underline{3+3} & 3+0-4 \\
\multicolumn{1}{l}{\qquad -4-4} & 3y^2+0xy-4x^2 \\
\multicolumn{1}{l}{\qquad \underline{-4-4}} &
\end{array}$$

(5.)

$$\begin{array}{l|l}
1-3-8+18-8 & \underline{1+2-2} \\
\underline{1+2-2} & 1-5+4 \\
-5-6+18 & a^2-5ab+4b^2 \\
\underline{-5-10+10} & \\
4+8-8 & \\
\underline{4+8-8} &
\end{array}$$

(6.)

$$\begin{array}{l|l}
1-5+10-10+5-1 & \underline{1-2+1} \\
\underline{1-2+1} & 1-3+3-1 \\
-3+9-10 & m^3-3m^2n+3mn^2-n^3 \\
\underline{-3+6-3} & \\
3-7+5 & \\
\underline{3-6+3} & \\
-1+2-1 & \\
\underline{-1+2-1} &
\end{array}$$

(7.)

$$\begin{array}{l|l}
1+0+0+0+0+0+0+1 & \underline{1+1} \\
\underline{1+1} & 1-1+1-1+1-1+1 \\
-1+0 & x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6 \\
\underline{-1-1} & \\
1+0 & \\
\underline{1+1} & \\
-1+0 & \\
\underline{-1-1} & \\
1+0 & \\
\underline{1+1} & \\
-1+0 & \\
\underline{-1-1} & \\
1+1 & \\
\underline{1+1} &
\end{array}$$

USEFUL FORMULAS.

(ART. **90**, pp. 46–48.)

5. Ans. $4m^2+12mn+9n^2$.

7. Ans. $16a^2b^2-8abx+x^2$.

10. Ans. $9a^4-b^6$.

16. $(c-a+b)(c+a-b)=(c-\overline{a-b})(c+\overline{a-b})$
$=c^2-(a-b)^2=c^2-a^2+2ab-b^2$.

17. $[(a+b)+(c+d)][(a+b)-(c+d)]$
$=(a+b)^2-(c+d)^2=a^2+2ab+b^2-c^2-2cd-d^2$.

FACTORING.

(ARTS. **101**, **102**, pp. 51–54.)

3. Ans. $(a+x)(a+x)$. 5. Ans. $3ab(b+3ad)$.

6. $1-8x^3=1^3-(2x)^3=(1-2x)(1+2x+4x^2)$, by 4.

7. $3bx^2+6bxy+3by^2=3b(x^2+2xy+y^2)$
$=3b(x+y)(x+y)$, by 1 and 2.

8. $81a^8-1=(9a^4)^2-1=(9a^4+1)(9a^4-1)$
$=(9a^4+1)(3a^2+1)(3a^2-1)$, by 3.

9. $5x^5+5y^5=5(x^5+y^5)$
$=5(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$, by 1 and 4.

10. $nx^2-2nxy+ny^2=n(x^2-2xy+y^2)$
$=n(x-y)(x-y)$.

12. $x^8-y^4=(x^4)^2-(y^2)^2=(x^4+y^2)(x^4-y^2)$
$=(x^4+y^2)(x^2+y)(x^2-y)$.

14. $c^2-(a-b)^2=[c-(a-b)][c+(a-b)]$
$=[c-a+b][c+a-b]$.

16. $am-an+bm-bn=a(m-n)+b(m-n)$
$=(a+b)(m-n)$.

17. $a^6-b^6=(a^3+b^3)\,(a^3-b^3)$
$=(a+b)\,(a^2-ab+b^2)\,(a-b)\,(a^2+ab+b^2).$

18. $x^2+12x+20=x^2+(10+2)\,x+10\times 2$
$=(x+10)\,(x+2).$

19. $a^2-16a+64=a^2-(8+8)\,a+8^2=(a-8)\,(a-8).$

20. $a^2+a-2=a^2+(2-1)\,a-1\times 2=(a-1)\,(a+2).$

21. $b^4-b^2-20=b^4+(4-5)\,b^2-5\times 4=(b^2+4)\,(b^2-5).$

22. $x^2y^2+12xy+27=x^2y^2+(9+3)\,xy+9\times 3$
$=(xy+9)\,(xy+3).$

24. $x^2-\frac{13}{4}x+\frac{3}{4}=x^2-(3+\frac{1}{4})\,x+3\times\frac{1}{4}=(x-3)\,(x-\frac{1}{4}).$

25. $x^2y-\frac{13}{3}xy-\frac{10}{3}y=y\,(x^2-\frac{13}{3}x-\frac{10}{3})$
$=y\,(x^2+[\frac{2}{3}-5]\,x-5\times\frac{2}{3})=y\,(x-5)\,(x+\frac{2}{3}).$

26. $5a^3b-5a^2b-30ab=5ab\,(a^2-a-6)$
$=5ab\,(a^2+[2-3]\,a-3\times 2)=5ab\,(a-3)\,(a+2).$

GREATEST COMMON DIVISOR.

CASE I.

(ART. **107**, pp. 56, 57.)

(2.)

$$\begin{array}{r l}
3ax^2-2a^2x & =ax\,(3x-2a) \\
a^2x^2-3abx & =ax\,(ax-3b) \\ \hline
\multicolumn{2}{c}{ax}
\end{array}$$

(3.)

$$\begin{array}{r l}
4a^2bc & =2\times 2a^2bc \\
6a^3bd & =2\times 3a^2bd \\
2a^4b^3c & =\quad\ 2a^4b^3c \\ \hline
 & \qquad 2a^2b
\end{array}$$

(4.)

$$\begin{array}{r l}
6x^3y-12x^2y^2+3xy^3 & =3xy\,(2x^2-4xy+y^2) \\
4ax^2+4axy+4a^2x & =4ax\,(x+y+a) \\ \hline
\multicolumn{2}{c}{\text{Ans. } x.}
\end{array}$$

(5.)

$$\begin{array}{r l}
x^3-a^3 & =(x-a)\,(x^2+ax+a^2) \\
x^2-a^2 & =(x-a)\,(x+a) \\ \hline
\multicolumn{2}{c}{x-a}
\end{array}$$

(6.)

$$m^2+2mn+n^2=(m+n)(m+n)$$
$$\underline{m^2-n^2=(m+n)(m-n)}$$
$$m+n$$

(7.)

$$5a^3-10a^2b+5ab^2=5a(a-b)(a-b)$$
$$\underline{8a^3-8a^2b=2^3a^2(a-b)}$$
$$a(a-b)$$

(8.)

$$b^2c+7bc+12c=c(b+3)(b+4)$$
$$\underline{3ab^2+21ab+36a=3a(b+3)(b+4)}$$
$$(b+3)(b+4)$$

(9.)

$$x^4-1=(x^2+1)(x+1)(x-1)$$
$$x^5+x^3=(x^2+1)x^3$$
$$\underline{x^4+2x^2+1=(x^2+1)(x^2+1}$$
$$x^2+1$$

CASE II.

(Art. **108**, pp. 57-60.)

(3.)

$$a^2-5ax+4x^2)\,a^3-a^2x+3ax^2-3x^3\,(a+4x$$
$$\underline{a^3-5a^2x+4ax^2}$$
$$4a^2x-ax^2-3x^3$$
$$\underline{4a^2x-20ax^2+16x^3}$$
$$19ax^2-19x^3$$
$$a-x)\,a^2-5ax+4x^2\,(a-4x$$
$$\underline{a^2-ax}$$
$$-4ax+4x^2$$
$$\underline{-4ax+4x^2}$$

(4.)

$$
\begin{array}{l}
x^2-7x+10\,)\,4x^3-25x^2+20x+25\,(\,4x+3\\
\qquad\qquad\qquad\;\underline{4x^3-28x^2+40x}\\
\qquad\qquad\qquad\qquad\quad 3x^2-20x+25\\
\qquad\qquad\qquad\qquad\quad \underline{3x^2-21x+30}\\
\qquad\qquad\qquad\qquad\qquad\qquad x-5\,)\,x^2-7x+10\,(\,x-2\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{x^2-5x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x+10\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2x+10}
\end{array}
$$

(5.)

$$
\begin{array}{l}
y^3-y^2x-yx^2+x^3\,)\,y^4-x^4\qquad\qquad(\,y+x\\
\qquad\qquad\qquad\;\;\underline{y^4-y^3x-y^2x^2+yx^3}\\
\qquad\qquad\qquad\qquad y^3x+y^2x^2-yx^3-x^4\\
\qquad\qquad\qquad\qquad \underline{y^3x-y^2x^2-yx^3+x^4}\\
\qquad\qquad\qquad\qquad\qquad 2y^2x^2-2x^4\\
\qquad\qquad\qquad\qquad\qquad\; y^2-x^2\,)\,y^3-y^2x-yx^2+x^3\,(\,y-x\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{y^3-yx^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -y^2x+x^3\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-y^2x+x^3}
\end{array}
$$

(6.)

$$
\begin{array}{l}
\begin{array}{r|r}
x^4+2x^3\qquad\quad-x-2 & x^4-x^3+2x^2+x+3\\
x^4-x^3+2x^2+x+3 & 3\\
\hline
\end{array}\\
\qquad 3x^3-2x^2-2x-5\,)\,3x^4-3x^3+6x^2+3x+9\,(\,x\\
\qquad\qquad\qquad\qquad\qquad\;\underline{3x^4-2x^3-2x^2-5x}\\
\qquad\qquad\qquad\qquad\qquad\quad -x^3+8x^2+8x+9\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{\qquad\qquad\qquad\qquad\;\;3}\\
\qquad\qquad\qquad\qquad\qquad\quad -3x^3+24x^2+24x+27\,(\,-1\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-3x^3+2x^2+2x+5}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 22x^2+22x+22\\
\qquad\qquad\qquad\qquad\qquad\quad x^2+x+1\,)\,3x^3-2x^2-2x-5\,(\,3x-5\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x^3+3x^2+3x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -5x^2-5x-5\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-5x^2-5x-5}
\end{array}
$$

(7.)

$$2a^4+3a^3x-9a^2x^2=a^2(2a^2+3ax-9x^2)$$

$$6a^4x-17a^3x^2+14a^2x^3-3ax^4=ax(6a^3-17a^2x+14ax^2-3x^3).$$

$$2a^2+3ax-9x^2)\,6a^3-17a^2x+14ax^2-3x^3\,(3a-13x$$

$$\underline{6a^3+9a^2x-27ax^2}$$

$$-26a^2x+41ax^2-3x^3$$

$$\underline{-26a^2x-39ax^2+117x^3}$$

$$80ax^2-120x^3$$

$$2a-3x)\,2a^2+3ax-9x^2\,(a+3x$$

$$\underline{2a^2-3ax}$$

$$6ax-9x^2$$

$$\underline{6ax-9x^2}$$

Ans. $a(2a-3x)=2a^2-3ax.$

(8.)

$$x^4+6x^3+11x^2+4x-4$$

$$\underline{x^4+2x^3-5x^2-12x-4}$$

$$4x^3+16x^2+16x$$

$$x^2+4x+4)\,x^4+2x^3-5x^2-12x-4\,(x^2-2x-1$$

$$\underline{x^4+4x^3+4x^2}$$

$$-2x^3-9x^2-12x$$

$$\underline{-2x^3-8x^2-8x}$$

$$-x^2-4x-4$$

$$\underline{-x^2-4x-4}$$

(9.)

$$6x^2y+4xy^2-2y^3=y(6x^2+4xy-2y^2)$$

$$8x^3+4x^2y-4xy^2=4x(2x^2+xy-y^2)$$

$$2x^2+xy-y^2)\,6x^2+4xy-2y^2\,(3$$

$$\underline{6x^2+3xy-3y^2}$$

$$xy+y^2$$

$$x+y)\,2x^2+xy-y^2\,(2x-y$$

$$\underline{2x^2+2xy}$$

$$-xy-y^2$$

$$\underline{-xy-y^2}$$

$2(x+y)=$ gr. com. factor.

(10.)

$$\begin{array}{r}
21x^3-26x^2+5x \\
\frac{2}{x} \\
\hline
6x^2-x-5\,)\,42x^2-52x+10\,(\,7 \\
\underline{42x^2-\ \ 7x-35} \\
-45x+45
\end{array}$$

$$\begin{array}{r}
x-\ 1\,)\,6x^2-\ \ x-5\,(\,6x+5 \\
\underline{6x^2-6x} \\
5x-5 \\
\underline{5x-5}
\end{array}$$

$$\begin{array}{r}
x-1\,)\,4x^3-12x^2+9x-1\,(\,4x^2-8x+1 \\
\underline{4x^3-\ \ 4x^2} \\
-\ \ 8x^2+9x \\
\underline{-\ \ 8x^2+8x} \\
x-1 \\
\underline{x-1}
\end{array}$$

LEAST COMMON MULTIPLE.

CASE I.

(Art. **113**, pp. 61, 62.)

(2.)

$$\begin{array}{l}
8a^4 \quad = 2^3a^4 \\
10a^3b = 2 \times 5a^3b \\
\underline{12a^2b^2 = 2^2 \times 3a^2b^2} \\
120a^4b^2 = 2^3 \times 3 \times 5a^4b^2
\end{array}$$

(3.)

$$\begin{array}{l}
2(a+b) = 2(a+b) \\
\underline{3(a^2-b^2) = 3(a+b)(a-b)} \\
6(a^2-b^2) = 2 \times 3(a+b)(a-b)
\end{array}$$

(4.)

$$\begin{array}{r}
(x+1)\ (x^2-1) = (x+1)^2(x-1) \\
\underline{x^3-1\ = (x-1)(x^2+x+1)} \\
(x+1)^2(x^3-1) = (x+1)^2(x-1)(x^2+x+1)
\end{array}$$

(5.)

$$\begin{array}{rl}
(a+b)^2 & = (a+b)^2 \\
a^2-b^2 & = (a+b)\ (a-b) \\
(a-b)^2 & = (a-b)^2 \\
a^3+3\,a^2\,b+3\,a\,b^2+b^3 & = (a+b)^3 \\
\hline
(a+b)\ (a^2-b^2)^2 & = (a+b)^3\,(a-b)^2
\end{array}$$

(6.)

$$\begin{array}{rl}
4\,(1-x^2) & = 2^2\,(1+x)\ (1-x) \\
8\,(1-x) & = 2^3\,(1-x) \\
8\,(1+x) & = 2^3\,(1+x) \\
4\,(1+x^2) & = 2^2\,(1+x^2) \\
\hline
8\,(1-x^4) & = 2^3\,(1+x)\ (1-x)\ (1+x^2)
\end{array}$$

CASE II.

(Art. **114**, pp. 62, 63.)

(2.)

$$x^3-2x^2y-xy^2+2y^3\,)\,3x^3-x^2y-3xy^2+y^3\,(\,3$$

$$\underline{3x^3-6x^2y-3xy^2+6y^3}$$

$$5x^2y-5y^3$$

$$x^2-y^2\,)\,x^3-2x^2y-xy^2+2y^3\,(\,x-2y$$

$$\underline{x^3\qquad -xy^2}$$

$$-2x^2y+2y^3$$

$$\underline{-2x^2y+2y^3}$$

$$\frac{(x^3-2\,x^2\,y-x\,y^2+2\,y^3)\ (3\,x^3-x^2\,y-3\,x\,y^2+y^3)}{x^2-y^2}=$$

$$(x-2y)\ (3\,x^3-x^2y-3\,xy^2+y^3)=(x-2\,y)\ \ (3\,x-y)\ (x^2-y^2)$$

(3.)

$$x^2-8x+7\,)\,x^3-9x^2+23x-15\,(\,x-1$$
$$\underline{x^3-8x^2+7x}$$
$$-x^2+16x-15$$
$$\underline{-x^2+8x-7}$$
$$8x-8$$
$$x-1\,)\,x^2-8x+7\,(\,x-7$$
$$\underline{x^2-x}$$
$$-7x+7$$
$$\underline{-7x+7}$$

$$\frac{(x^3-9x^2+23x-15)\,(x^2-8x+7)}{x-1}=(x^3-9x^2+23x-15)\,(x-7)$$

(4.)

$$2x^2+3x-2\,)\,6x^2-x-1\,(\,3$$
$$\underline{6x^2+9x-6}$$
$$-10x+5$$
$$2x-1\,)\,2x^2+3x-2\,(\,x+2$$
$$\underline{2x^2-x}$$
$$4x-2$$
$$\underline{4x-2}$$

$$2x-1\,)\,6x^2-x-1\,(\,3x+1$$
$$\underline{6x^2-3x}$$
$$2x-1$$
$$\underline{2x-1}$$

$$\frac{(2x^2+3x-2)\,(6x^2-x-1)}{2x-1}=(2x^2+3x-2)\,(3x+1)$$

(5.)

FIRST SOLUTION.

$$x^2+2x-8\,)\,x^2+5x+4\,(\,1$$
$$\underline{x^2+2x-8}$$
$$3x+12$$
$$x+4\,)\,x^2+2x-8\,(\,x-2$$
$$\underline{x^2+4x}$$
$$-2x-8$$
$$\underline{-2x-8}$$

$$\frac{(x^2+2x-8)\,(x^2+5x+4)}{x+4}=(x-2)\,(x^2+5x+4)$$

$$=x^3+3x^2-6x-8$$

$$\begin{array}{l}
x^2+7x+12\,)\,x^3+3x^2-6x-8\,(\,x-4\\
\qquad\qquad\quad \underline{x^3+7x^2+12x}\\
\qquad\qquad\qquad -4x^2-18x-8\\
\qquad\qquad\qquad \underline{-4x^2-28x-48}\\
\qquad\qquad\qquad\qquad\quad 10x+40\\
\qquad\qquad\qquad\qquad\qquad x+4\,)\,x^2+7x+12\,(\,x+3\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+4x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x+12\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x+12}
\end{array}$$

$$\frac{(x^3+3x^2-6x-8)\,(x^2+7x+12)}{x+4}=(x^3+3x^2-6x-8)\,(x+3)$$

$$=x^4+6x^3+3x^2-26x-24$$

SECOND SOLUTION.

$$\left.\begin{array}{r}
x^2+5x+4=(x+4)\,(x+1)\\
x^2+2x-8=(x+4)\,(x-2)\\
x^2+7x+12=(x+4)\,(x+3)
\end{array}\right\}\text{Art. 102.}$$

$$\overline{x^4+6x^3+3x^2-26x-24=(x+4)\,(x+1)\,(x-2)\,(x+3)}$$

(6.)

FIRST SOLUTION.

$$\begin{array}{l}
x^2-4a^2\,)\,x^3+2ax^2+4a^2x+8a^3\,(\,x+2a\\
\qquad\qquad \underline{x^3\qquad\quad -4a^2x}\\
\qquad\qquad\quad 2ax^2+8a^2x+8a^3\\
\qquad\qquad\quad \underline{2ax^2\qquad\quad -8a^3}\\
\qquad\qquad\qquad\qquad 8a^2x+16a^3\\
\qquad\qquad\qquad\qquad\quad x+2a\,)\,x^2-4a^2\,(\,x-2a\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2+2ax}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2ax-4a^2\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2ax-4a^2}
\end{array}$$

$$\frac{(x^2-4a^2)(x^3+2ax^2+4a^2x+8a^3)}{x+2a}$$

$$=(x-2a)(x^3+2ax^2+4a^2x+8a^3)=x^4-16a^4.$$

$$\begin{array}{r|l} x^3-2ax^2+4a^2x-8a^3)\; x^4-16a^4 & (x+2a \\ \underline{x^4-2ax^3+4a^2x^2-8a^3x} & \\ 2ax^3-4a^2x^2+8a^3x-16a^4 & \\ \underline{2ax^3-4a^2x^2+8a^3x-16a^4} & \end{array}$$

$$\frac{(x^4-16a^4)(x^3-2ax^2+4a^2x-8a^3)}{x^3-2ax^2+4a^2x-8a^3}=x^4-16a^4.$$

SECOND SOLUTION.

$$\begin{aligned} x^2-4a^2&=(x-2a)(x+2a) \\ x^3+2ax^2+4a^2x+8a^3&=(x+2a)(x^2+4a^2) \\ x^3-2ax^2+4a^2x-8a^3&=(x-2a)(x^2+4a^2) \\ \hline x^4-16a^4&=(x-2a)(x+2a)(x^2+4a^2) \end{aligned}$$

FRACTIONS.

REDUCTION OF FRACTIONS.

CASE I.

(Art. **129**, pp. 67–69.)

3. $\dfrac{56\,b\,m^2x^6}{76\,b\,m\,x^2}=\dfrac{14\,m\,x^4\times4\,b\,m\,x^2}{19\times4\,b\,m\,x^2}=\dfrac{14\,m\,x^4}{19}.$

4. $\dfrac{19\,a\,b^2c\,d\,e^3}{76\,a^3b\,c\,d\,e^5}=\dfrac{b\times19\,a\,b\,c\,d\,e^3}{4\,a^2e^2\times19\,a\,b\,c\,d\,e^3}=\dfrac{b}{4\,a^2e^2}.$

5. $\dfrac{6\,a^2+5\,a\,x-6\,x^2}{6\,a^2+13\,a\,x+6\,x^2}=\dfrac{(3\,a-2\,x)(2\,a+3\,x)}{(3\,a+2\,x)(2\,a+3\,x)}=\dfrac{3\,a-2\,x}{3\,a+2\,x}.$

6. $\dfrac{a^2-x^2}{a^4-x^4}=\dfrac{a^2-x^2}{(a^2-x^2)(a^2+x^2)}=\dfrac{1}{a^2+x^2}.$

7. $\dfrac{x^6-y^6}{x^4-y^4}=\dfrac{(x^4+x^2y^2+y^4)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\dfrac{x^4+x^2y^2+y^4}{x^2+y^2}.$

8. $\frac{2x^3-16x-6}{3x^3-24x-9}=\frac{2(x^3-8x-3)}{3(x^3-8x-3)}=\frac{2}{3}.$

9. $\frac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}=\frac{(a+x)(a^2+bx)}{(a^2-bx)(a^2+bx)}=\frac{a+x}{a^2-bx}.$

10. $\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}=\frac{x^m(a-bx)}{bx(a+bx)(a-bx)}=\frac{x^m}{bx(a+bx)}$

$=\frac{x^{m-1}}{b(a+bx)}.$

11. $\frac{bx+2}{2b+(b^2-4)x-2bx^2}=\frac{bx+2}{(b-2x)(2+bx)}=\frac{1}{b-2x}.$

CASE II.

(Arts. 130, 131, pp. 69–71.)

2. $\frac{x^3+y^3}{x+y}=\frac{(x+y)(x^2-xy+y^2)}{x+y}=x^2-xy+y^2.$

3. Ans. $1+ax.$

4. $\frac{10x^2-5x+3}{5x}=(10x^2-5x+3)\div 5x=2x-1+\frac{3}{5x}.$

5. $\frac{a^3-x^3+b^3}{a-x}=\frac{(a-x)(a^2+ax+x^2)+b^3}{a-x}$

$=a^2+ax+x^2+\frac{b^3}{a-x}.$

6.
$$\begin{array}{l}
x-3\,)\;2x^2+5\quad\left(2x+6+\frac{23}{x-3}\right.\\
\qquad\quad\underline{2x^2-6x}\\
\qquad\qquad\quad 6x+5\\
\qquad\qquad\quad \underline{6x-18}\\
\qquad\qquad\qquad\;\; 23
\end{array}$$

7. $\frac{x^3-1}{x-1}=\frac{(x-1)(x^2+x+1)}{x-1}=x^2+x+1.$

8.
$$\begin{array}{l}
2x^2-x+1\,)\;4x^2-2x\qquad\left(2-\frac{2}{2x^2-x+1}\right.\\
\qquad\qquad\qquad\underline{4x^2-2x+2}\\
\qquad\qquad\qquad\qquad\quad -2
\end{array}$$

9. $\frac{m^2(m^4-n^4)+3(m^3-mn^2)}{m^2(m^2-n^2)}=\frac{m^2(m^2-n^2)(m^2+n^2)+3m(m^2-n^2)}{m^2(m^2-n^2)}$

$=\frac{m^2(m^2+n^2)+3m}{m^2}=m^2+n^2+\frac{3}{m}.$

11. $\frac{7x^2y^3}{4x^3z^2}=\frac{7y^3}{4xz^2}=7y^3(4^{-1}x^{-1}z^{-2})=7(4^{-1}x^{-1}y^3z^{-2}).$

12. $\frac{x^3y^3}{a^{-2}b^{-1}x^{-1}y}=a^2bxx^3y^2y^{-1}=a^2bx^4y.$

13. Ans. $x^3y^2z.$

14. $\frac{x^2(a+b)}{(a-b)^{-2}}=x^2(a+b)(a-b)^2=x^2(a^2-b^2)(a-b)$

$=x^2(a^3-a^2b-ab^2+b^3).$

15. $\frac{5x(x-1)^{-1}+3(x-1)^{-1}}{(x-1)^{-2}}=5x(x-1)+3(x-1)$

$=5x^2-2x-3.$

CASE III.

(ART. 132, pp. 71, 72.)

2. $a+\frac{b^2-cd}{n}=\frac{an+b^2-cd}{n}.$

3. $7x-\frac{4n^2+5a}{8}=\frac{56x-(4n^2+5a)}{8}=\frac{56x-4n^2-5a}{8}.$

4. Ans. $\frac{mx-a+b}{m}.$

5. $a+b-\frac{a^2+b^2}{a+b}=\frac{a^2+2ab+b^2-a^2-b^2}{a+b}=\frac{2ab}{a+b}.$

6. $1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}.$

7. $2a-\frac{3a^2-2b^2}{2a}=\frac{4a^2-3a^2+2b^2}{2a}=\frac{a^2+2b^2}{2a}.$

8. $x+1+\frac{x+1}{x}=\frac{x^2+x+x+1}{x}=\frac{(x+1)^2}{x}.$

9. $a-b-\frac{a^2+b^2}{a+b}=\frac{a^2-b^2-a^2-b^2}{a+b}=\frac{-2b^2}{a+b}.$

10. $x^2-3x-\frac{3x(3-x)}{x-2}=\frac{x^3-5x^2+6x-9x+3x^2}{x-2}$

$=\frac{x^3-2x^2-3x}{x-2}=\frac{x(x^2-2x-3)}{x-2}.$

CASE IV.

(ARTS. **133**, **134**, pp. 72–75.)

2. $\frac{3x}{y}=\frac{3x\times x(b-c)}{y\times x(b-c)}=\frac{3bx^2-3cx^2}{bxy-cxy}.$

$\frac{4m}{x}=\frac{4m\times y(b-c)}{x\times y(b-c)}=\frac{4bmy-4cmy}{bxy-cxy}.$

$\frac{a}{b-c}=\frac{a\times xy}{(b-c)\times xy}=\frac{axy}{bxy-cxy}.$

3. $a=\frac{ab}{b}$; $b=\frac{b^2}{b}$; $c=\frac{bc}{b}$; $d=\frac{bd}{b}$.

4. The least common multiple of $6a$ and 4 is $12a$.

$12a\div 6a=2$; $\frac{(2x+1)\times 2}{6a\times 2}=\frac{2(2x+1)}{12a}.$

$12a\div 4=3a$; $\frac{5x\times 3a}{4\times 3a}=\frac{15ax}{12a}.$

5. $\frac{-5y}{7bx}=\frac{-5y\times(-5d)}{7bx\times(-5d)}=\frac{25dy}{-35bdx}.$

$\frac{3c}{d}=\frac{3c\times(-5)\times 7bx}{d\times(-5)\times 7bx}=\frac{-105bcx}{-35bdx}.$

$\frac{4}{-5}=\frac{4\times 7bxd}{-5\times 7bxd}=\frac{28bdx}{-35bdx}.$

6. The least common multiple of $a+b$, $a-b$, and a^2-b^2, is a^2-b^2.

$(a^2-b^2)\div(a+b)=a-b$; $\frac{a^2\times(a-b)}{(a+b)\times(a-b)}$
$=\frac{a^2(a-b)}{a^2-b^2}.$

$(a^2-b^2)\div(a-b)=a+b$; $\frac{(a+b)\times(a+b)}{(a-b)\times(a+b)}=\frac{(a+b)^2}{a^2-b^2}.$

$(a^2-b^2)\div(a^2-b^2)=1$; $\frac{a^2b^2\times 1}{a^2-b^2\times 1}=\frac{a^2b^2}{a^2-b^2}.$

7. The least common multiple of $1-x$, $(1-x)^2$, and $(1-x)^3$ is $(1-x)^3$.

$(1-x)^3 \div (1-x) = (1-x)^2$; $\dfrac{a\,y \times (1-x)^2}{(1-x) \times (1-x)^2} = \dfrac{a\,y\,(1-x)^2}{(1-x)^3}$.

$(1-x)^3 \div (1-x)^2 = 1-x$; $\dfrac{a\,x^2 \times (1-x)}{(1-x)^2 \times (1-x)} = \dfrac{a\,x^2\,(1-x)}{(1-x)^3}$.

$(1-x)^3 \div (1-x)^3 = 1$; $\dfrac{x\,y^2 \times 1}{(1-x)^3 \times 1} = \dfrac{x\,y^2}{(1-x)^3}$.

9. $(a^3+b^3) \div (a^2-ab+b^2) = a+b$; $\dfrac{(a-b) \times (a+b)}{(a^2-ab+b^2) \times (a+b)} = \dfrac{a^2-b^2}{a^3+b^3}$.

10. $10\,a^2 b \div 5 = 2\,a^2 b$; $\dfrac{2\,d \times 2\,a^2 b}{5 \times 2\,a^2 b} = \dfrac{4\,a^2 b\,d}{10\,a^2 b}$.

$10\,a^2 b \div a\,b = 10\,a$; $\dfrac{3\,c^2 \times 10\,a}{a\,b \times 10\,a} = \dfrac{30\,a\,c^2}{10\,a^2 b}$.

11. $(a^2-b^2) \div 1 = a^2-b^2$; $\dfrac{a\,b \times (a^2-b^2)}{1 \times (a^2-b^2)} = \dfrac{a\,b\,(a^2-b^2)}{a^2-b^2}$.

$(a^2-b^2) \div (a+b) = a-b$; $\dfrac{(a-b) \times (a-b)}{(a+b) \times (a-b)} = \dfrac{(a-b)^2}{a^2-b^2}$.

$(a^2-b^2) \div (a-b) = a+b$; $\dfrac{(a+b) \times (a+b)}{(a-b) \times (a+b)} = \dfrac{(a+b)^2}{a^2-b^2}$.

ADDITION OF FRACTIONS.

(ART. 135, pp. 75–77.)

2. $\dfrac{4\,a}{7} + \dfrac{a-3}{4} = \dfrac{16\,a}{28} + \dfrac{7\,a-21}{28} = \dfrac{23\,a-21}{28}$.

3. $4\,m + \dfrac{3\,a-1}{2} + \dfrac{4\,n+2}{3} = \dfrac{24\,m}{6} + \dfrac{9\,a-3}{6} + \dfrac{8\,n+4}{6}$
$= \dfrac{24\,m+9\,a+8\,n+1}{6}$.

4. $\frac{4a-3}{5}+\frac{7a+1}{3}+\frac{3a}{2}=\frac{24a-18}{30}+\frac{70a+10}{30}+\frac{45a}{30}$

$=\frac{139a-8}{30}$, Ans.

5. $\frac{3}{a+b}+\frac{3}{a-b}=\frac{3a-3b}{a^2-b^2}+\frac{3a+3b}{a^2-b^2}=\frac{6a}{a^2-b^2}.$

6. $a-\frac{3x^2}{b}+b+\frac{2ax}{c}=a+b-\frac{3cx^2}{bc}+\frac{2abx}{bc}$

$=a+b+\frac{2abx-3cx^2}{bc}.$

7. $2+\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{2a^2-2b^2}{a^2-b^2}+\frac{a^2+2ab+b^2}{a^2-b^2}$

$+\frac{a^2-2ab+b^2}{a^2-b^2}=\frac{4a^2}{a^2-b^2}.$

8. $\frac{1+x}{1+x+x^2}+\frac{1-x}{1-x+x^2}=\frac{1+x^3}{1+x^2+x^4}+\frac{1-x^3}{1+x^2+x^4}$

$=\frac{2}{1+x^2+x^4}.$

9. $\frac{3}{a}-\frac{a+1}{b-1}-\frac{a-1}{b+1}=\frac{3b^2-3}{ab^2-a}-\frac{a^2b+ab+a^2+a}{ab^2-a}$

$-\frac{a^2b-ab-a^2+a}{ab^2-a}=\frac{3b^2-2a^2b-2a-3}{ab^2-a}.$

10. $2a+3a+a+\frac{2x}{5}-\frac{8x}{9}=6a+\frac{18x}{45}-\frac{40x}{45}$

$=6a-\frac{22x}{45}.$

11. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$

$=\frac{(a+b)(a-b)}{(a-b)(b-c)(c-a)}+\frac{(b+c)(b-c)}{(a-b)(b-c)(c-a)}$

$+\frac{(c+a)(c-a)}{(a-b)(b-c)(c-a)}=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(a-b)(b-c)(c-a)}$

$=\frac{0}{(a-b)(b-c)(c-a)}=0.$

12. $$\frac{1}{a(a-b)(a-c)} = \frac{bc(b-c)}{abc(a-b)(a-c)(b-c)}.$$

$$\frac{1}{b(b-c)(b-a)} = \frac{-ac(a-c)}{abc(a-b)(a-c)(b-c)}.$$

$$\frac{1}{c(c-a)(c-b)} = \frac{ab(a-b)}{abc(a-b)(a-c)(b-c)}.$$

Adding these three fractions, we obtain

$$\frac{b^2c - bc^2 - a^2c + ac^2 + a^2b - ab^2}{abc(a-b)(a-c)(b-c)}$$

$$= \frac{a^2b - ab^2 + b^2c - bc^2 - a^2c + ac^2}{abc(a^2b - ab^2 + b^2c - bc^2 - a^2c + ac^2)} = \frac{1}{abc}.$$

SUBTRACTION OF FRACTIONS.

(Art. 136, pp. 77, 78.)

2. $$\frac{7x}{5} - \frac{4x}{7} = \frac{49x}{35} - \frac{20x}{35} = \frac{29x}{35}.$$

3. $$\frac{2}{a-1} - \frac{2}{a+1} = \frac{2a+2}{a^2-1} - \frac{2a-2}{a^2-1} = \frac{4}{a^2-1}.$$

4. $$\frac{3a-2b}{3c} - \frac{2a-4b}{5b} = \frac{15ab-10b^2}{15bc} - \frac{6ac-12bc}{15bc}$$

$$= \frac{15ab - 6ac - 10b^2 + 12bc}{15bc}, \text{ Ans.}$$

5. $$\frac{x+y}{x-y} - \frac{x-y}{x+y} = \frac{x^2+2xy+y^2}{x^2-y^2} - \frac{x^2-2xy+y^2}{x^2-y^2} = \frac{4xy}{x^2-y^2}.$$

6. $$3a - a + \frac{a+b}{d} + \frac{a-b}{d} = 2a + \frac{2a}{d}.$$

7. $$7x - x + \frac{12a-3b}{6} - \frac{6a-4b}{6} = 6x + \frac{6a+b}{6}$$

$$= 6x + a + \frac{b}{6}.$$

8. $$\frac{a^2+2ab+b^2}{ab} - \frac{a^2-2ab+b^2}{ab} = \frac{4ab}{ab} = 4.$$

9. $$\frac{a}{b} + \frac{b}{a} + 2 = \frac{(a+b)^2}{ab} = \frac{(a+b)^2(a-b)^2}{ab(a-b)^2} = \frac{(a^2-b^2)^2}{ab(a-b)^2};$$

$$\frac{(a^2+b^2)^2}{ab\,(a-b)^2}-\frac{(a^2-b^2)^2}{ab\,(a-b)^2}=\frac{a^4+2\,a^2b^2+b^4}{ab\,(a-b)^2}$$

$$-\frac{a^4-2\,a^2b^2+b^4}{ab\,(a-b)^2}=\frac{4\,a^2b^2}{ab\,(a-b)^2}=\frac{4\,ab}{(a-b)^2}.$$

10. $$\frac{5}{2\,(x+1)}-\frac{1}{10\,(x-1)}-\frac{24}{5\,(2\,x+3)}$$

$$=\frac{25\,(x-1)\,(2\,x+3)}{10\,(x^2-1)\,(2\,x+3)}-\frac{(x+1)\,(2\,x+3)}{10\,(x^2-1)\,(2\,x+3)}$$

$$-\frac{48\,(x^2-1)}{10\,(x^2-1)\,(2\,x+3)}$$

$$=\frac{50\,x^2+25\,x-75-(2\,x^2+5\,x+3)-(48\,x^2-48)}{10\,(x^2-1)\,(2\,x+3)}$$

$$=\frac{20\,x-30}{10\,(x^2-1)\,(2\,x+3)}=\frac{2\,x-3}{(x^2-1)\,(2\,x+3)}.$$

MULTIPLICATION OF FRACTIONS.

(Art. **137**, pp. 79, 80.)

4. $\dfrac{3\,abx^2}{5\,ay^2}\times\dfrac{5\,xy^2}{3\,abx^3}=\dfrac{15\,abx^3y^2}{15\,a^2bx^3y^2}=\dfrac{1}{a}$, Ans.

5. $\dfrac{m+n}{x+y}\times\dfrac{m+n}{x+y}=\dfrac{(m+n)^2}{(x+y)^2}=\dfrac{m^2+2\,mn+n^2}{x^2+2\,xy+y^2}.$

6. $\left(a+\dfrac{hy}{a}\right)\times\dfrac{m}{n}=\dfrac{a^2+hy}{a}\times\dfrac{m}{n}=\dfrac{a^2m+hmy}{an}.$

8. $\dfrac{4\,x+2}{3}\times\dfrac{5\,x}{2\,x+1}=\dfrac{2\,(2\,x+1)}{3}\times\dfrac{5\,x}{2\,x+1}=\dfrac{10x}{3}.$

9. $\dfrac{2\,x-1}{x+2}\times\dfrac{3\,x+1}{x-3}=\dfrac{(2\,x-1)\,(3\,x+1)}{(x+2)\,(x-3)}=\dfrac{6\,x^2-x-1}{x^2-x-6}.$

11. Cancel the common factor $(a-b)$.

(12.)

FIRST SOLUTION.

$$\begin{array}{l}
x^2 - x + 1 \\
\underline{1 + \frac{1}{x} + \frac{1}{x^2}} \\
x^2 - x + 1 \\
\qquad x - 1 + \frac{1}{x} \\
\qquad\qquad \underline{1 - \frac{1}{x} + \frac{1}{x^2}} \\
x^2 \qquad + 1 \qquad + \frac{1}{x^2}
\end{array}$$

SECOND SOLUTION.

$$\frac{1}{x^2} + \frac{1}{x} + 1 = \frac{1 + x + x^2}{x^2}$$

$$\frac{(1 + x + x^2)(x^2 - x + 1)}{x^2} = \frac{x^4 + x^2 + 1}{x^2}$$

$$= x^2 + 1 + \frac{1}{x^2}$$

13. $\frac{x^n}{y^m} \times \frac{y^n}{x^m} = \frac{x^n y^n}{x^m y^m} = \frac{x^{n-m}}{y^{m-n}}.$

14. $\frac{1 - x^2}{1 + y} \times \frac{1 - y^2}{x + x^2} \times \left(1 + \frac{x}{1 - x}\right)$

$$= \frac{(1 + x)(1 - x)}{1 + y} \times \frac{(1 + y)(1 - y)}{x(1 + x)} \times \frac{1}{1 - x} = \frac{1 - y}{x}.$$

15. $\frac{3ab^2}{a - b} \times (a^2 - b^2) = \frac{3ab^2(a + b)(a - b)}{a - b} = 3ab^2(a + b).$

16. $b\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{b} + \frac{b}{a}\right) = b\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) = \frac{b(a^4 - b^4)}{a^2 b^2} = \frac{a^4 - b^4}{a^2 b}.$

DIVISION OF FRACTIONS.

(Arts. **138–140**, pp. 80–83.)

2. $\frac{7m^2}{2} \div \frac{3n^2}{13} = \frac{7m^2}{2} \times \frac{13}{3n^2} = \frac{91m^2}{6n^2}.$

3. $\frac{1}{xy} \div xy = \frac{1}{xy} \times \frac{1}{xy} = \frac{1}{x^2 y^2}.$

4. $\frac{3x + 1}{9} \div \frac{4x}{3} = \frac{3x + 1}{9} \times \frac{3}{4x} = \frac{3x + 1}{3} \times \frac{1}{4x}$

$$= \frac{3x + 1}{12x}.$$

5. $\frac{4x}{2x-1} \div \frac{2}{x+1} = \frac{4x}{2x-1} \times \frac{x+1}{2} = \frac{2x}{2x-1} \times \frac{x+1}{1}$
$= \frac{2x^2+2x}{2x-1}$, Ans.

6. $\frac{2a+b}{3a-2b} \div \frac{3a+2b}{4a+b} = \frac{2a+b}{3a-2b} \times \frac{4a+b}{3a+2b}$
$= \frac{8a^2+6ab+b^2}{9a^2-4b^2}$.

7. $\frac{x^2-y^2}{x^2-2xy+y^2} \div \frac{x^2+xy}{x-y} = \frac{(x+y)(x-y)}{(x-y)(x-y)} \times \frac{x-y}{x(x+y)}$
$= \frac{1}{x}$.

8. $\frac{9x^2-4y^2}{x+y} \div \frac{3x-2y}{x^2-y^2} = \frac{(3x+2y)(3x-2y)}{x+y}$
$\times \frac{(x+y)(x-y)}{3x-2y} = (3x+2y)(x-y)$
$= 3x^2-xy-2y^2$.

9. $\frac{2a-b}{4ac} \div \frac{2ac}{3b-1} = \frac{2a-b}{4ac} \times \frac{3b-1}{2ac}$
$= \frac{6ab-2a-3b^2+b}{8a^2c^2}$.

10. $\left(\frac{a}{a+b} + \frac{b}{a-b}\right) \div \left(\frac{a}{a-b} - \frac{b}{a+b}\right) = \frac{a^2+b^2}{a^2-b^2} \div \frac{a^2+b^2}{a^2-b^2}$
$= \frac{a^2+b^2}{a^2-b^2} \times \frac{a^2-b^2}{a^2+b^2} = 1$.

11. $\left(x^4 - \frac{1}{x^4}\right) \div \left(x + \frac{1}{x}\right) = \frac{x^8-1}{x^4} \div \frac{x^2+1}{x} = \frac{x^8-1}{x^4}$
$\times \frac{x}{x^2+1} = \frac{(x^2+1)(x^6-x^4+x^2-1)}{x^4} \times \frac{x}{x^2+1}$
$= \frac{x^6-x^4+x^2-1}{x^3} = x^3 - x + \frac{1}{x} - \frac{1}{x^3}$.

The division may also be performed without reducing the mixed quantities to fractional forms, as follows:

$$\begin{array}{l|l}
x^4-\frac{1}{x^4} & x+\frac{1}{x} \\ \cline{2-2}
x^4+x^2 & x^3-x+\frac{1}{x}-\frac{1}{x^3} \\ \cline{1-1}
\quad -x^2-\frac{1}{x^4} & \\
\quad -x^2-1 & \\ \cline{1-1}
\qquad 1-\frac{1}{x^4} & \\
\qquad 1+\frac{1}{x^2} & \\ \cline{1-1}
\qquad\quad -\frac{1}{x^2}-\frac{1}{x^4} & \\
\qquad\quad -\frac{1}{x^2}-\frac{1}{x^4} & \\ \cline{1-1}
\end{array}
\qquad
\begin{array}{l|l}
x^4-x^{-4} & x+x^{-1} \\ \cline{2-2}
x^4+x^2 & x^3-x+x^{-1}-x^{-3} \\ \cline{1-1}
\quad -x^2-x^{-4} & \\
\quad -x^2-x^0 & \\ \cline{1-1}
\qquad x^0-x^{-4} & \\
\qquad x^0+x^{-2} & \\ \cline{1-1}
\qquad\quad -x^{-2}-x^{-4} & \\
\qquad\quad -x^{-2}-x^{-4} & \\ \cline{1-1}
\end{array}$$

12. $\frac{c-b}{c+b}-\frac{c^3-b^3}{c^3+b^3}=\frac{c^3-2bc^2+2b^2c-b^3}{c^3+b^3}-\frac{c^3-b^3}{c^3+b^3}$
$=\frac{2(b^2c-bc^2)}{c^3+b^3}$; $\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}=\frac{c^2+2bc+b^2}{c^2-b^2}+\frac{c^2+b^2}{c^2-b^2}$
$=\frac{2(c^2+bc+b^2)}{c^2-b^2}$; $\frac{2(b^2c-bc^2)}{c^3+b^3}\div\frac{2(c^2+bc+b^2)}{c^2-b^2}$
$=\frac{2(b^2c-bc^2)}{c^3+b^3}\times\frac{c^2-b^2}{2(c^2+bc+b^2)}=\frac{bc(b-c)}{c^2-bc+b^2}$
$\times\frac{c-b}{c^2+bc+b^2}=-\frac{bc(b-c)^2}{b^4+b^2c^2+c^4}$.

14. $\frac{\frac{a}{b}}{m+n}=\frac{\frac{a}{b}\times b}{(m+n)b}=\frac{a}{bm+bn}$.

15. $\frac{a+\frac{b}{c}}{x-\frac{m}{n}}=\frac{\left(a+\frac{b}{c}\right)cn}{\left(x-\frac{m}{n}\right)cn}=\frac{acn+bn}{cnx-cm}$.

16. $\frac{m-\frac{n}{3}}{x}=\frac{\left(m-\frac{n}{3}\right)\times 3}{x\times 3}=\frac{3m-n}{3x}$.

17. $\frac{y-x+\frac{a}{2}}{7\frac{3}{4}}=\frac{\left(y-x+\frac{a}{2}\right)\times 4}{7\frac{3}{4}\times 4}=\frac{4y-4x+2a}{31}$

18. $$1+\frac{x+1}{3-x}=\frac{3-x}{3-x}+\frac{x+1}{3-x}=\frac{4}{3-x};\quad \frac{1}{x+\frac{1}{1+\frac{x+1}{3-x}}}$$

$$=\frac{1}{x+\frac{3-x}{4}}=\frac{1\times 4}{(x+\frac{3-x}{4})\times 4}=\frac{4}{4x+3-x}=\frac{4}{3(x+1)}.$$

19. $$\frac{x^2+y^{-3}}{m n^{-4} e^{-2}}=\frac{(x^2+y^{-3})\times e^2 n^4 y^3}{m n^{-4} e^{-2}\times e^2 n^4 y^3}=\frac{e^2 n^4 x^2 y^3+e^2 n^4}{m y^3}.$$

20. $$\frac{1-a^{-2}-y^2}{1-x^{-3}y^{-2}+x^{-2}}=\frac{(1-a^{-2}-y^2)\times a^2 x^3 y^2}{(1-x^{-3}y^{-2}+x^{-2})\times a^2 x^3 y^2}$$

$$=\frac{a^2 x^3 y^2-x^3 y^2-a^2 x^3 y^4}{a^2 x^3 y^2-a^2+a^2 x y^2}.$$

SIMPLE EQUATIONS.

TRANSFORMATION OF EQUATIONS.

(Art. 151, pp. 86, 87.)

4. Ans. $4x+12x=25-9$.

(Art. 152, pp. 87, 88.)

2. Multiply by 6.
3. Multiply by ben.
4. Multiply by bc. Ans. $abx+b^2-ac=cx+d$.
5. Multiply by 24.
6. Multiply by bde.
7. Multiply by 16.

SOLUTION OF EQUATIONS.

(Art. 155, pp. 89–92.)

5. Given $12x-3x-2x=63$

Uniting terms, $7x=63$

Dividing by 7, $x=9$

6 Given $x-\frac{x}{4}+13=\frac{x}{2}+40$

Multiplying by 4, $4x-x+52=2x+160$

Transposing, $4x-x-2x=160-52$

Uniting, $x=108$

7. Given $7x+2x=12x-36$

Transposing, $7x+2x-12x=-36$

Uniting, $-3x=-36$

Dividing by -3, $x=12$, Ans.

8. Given $\frac{x}{5}+\frac{x}{12}=\frac{x}{10}+22$

Multiplying by 60, $12x+5x=6x+1320$

Transposing, $12x+5x-6x=1320$

Uniting, $11x=1320$

Dividing by 11, $x=120$

9. Given $\frac{a}{x}=b+c$

Multiplying by x, $a=bx+cx$

Transposing, $-bx-cx=-a$

Ch. signs (Art. 151), $bx+cx=a$

Factoring, $(b+c)x=a$

Dividing by $b+c$, $x=\frac{a}{b+c}$

10. Given $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26$

Mult. by 28, $28x-4x+560=14x+7x+728$

Transposing, $28x-4x-14x-7x=728-560$

Uniting, $3x=168$

Dividing by 3, $x=56$

11. Given $3x+\frac{3x}{4}+15=\frac{x}{2}+41$

Multiplying by 4, $12x+3x+60=2x+164$

Transposing, $12x+3x-2x=164-60$
Uniting, $13x=104$
Dividing by 13, $x=8$

12. Given $x-\frac{4x+8}{6}=8$
Multiplying by 6, $6x-4x-8=48$
Transposing and uniting, $2x=56$
Dividing by 2, $x=28$

13. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$
Mult. by 16, $336+3x-11=10x-10+776-56x$
Transposing, $3x-10x+56x=776-10+11-336$
Uniting, $49x=441$
Dividing by 49, $x=9$, Ans.

14. Given $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5$
Multiplying by 30,
$510x-50x+40-48x-24=600x-45x-120-150$
Transposing,
$510x-50x-48x-600x+45x=-120-150+24-40$
Uniting and ch. signs, $143x=286$
Dividing by 143, $x=2$

15. Given $ax^2+bx=mx^2+nx$
Dividing by x, $ax+b=mx+n$
Transposing, $ax-mx=n-b$
Dividing by $a-m$, $x=\frac{n-b}{a-m}$

16.

FIRST SOLUTION.

Given $a+x^{-1}=b+c+dx^{-1}$
Multiplying by x, $ax+1=bx+cx+d$

Transposing, $ax-bx-cx=d-1$

Dividing by $a-b-c$, $x=\dfrac{d-1}{a-b-c}$

SECOND SOLUTION.

Given $a+x^{-1}=b+c+dx^{-1}$

Transposing, $x^{-1}-dx^{-1}=b+c-a$

Dividing by $1-d$, $x^{-1}=\dfrac{b+c-a}{1-d}$

Taking reciprocals, $x=\dfrac{1-d}{b+c-a}$

Changing signs, $x=\dfrac{d-1}{a-b-c}$

17. Given $\dfrac{2(4x+3)}{x+3}+\dfrac{3}{x+1}=8$

Clearing of fractions,

$$8x^2+14x+6+3x+9=8x^2+32x+24$$

Transposing and uniting, $-15x=9$

Dividing by -15, $x=-\frac{3}{5}$

18. Given $\dfrac{x}{4}-4=\dfrac{5x}{3}-\dfrac{7}{6}$

Multiplying by 12, $3x-48=20x-14$

Transposing and uniting, $-17x=34$

Dividing by -2, $x=-2$

19. Given $ax+b=\dfrac{x}{a}+b^{-1}$

Multiplying by ab, $a^2bx+ab^2=bx+a$

Transposing, $a^2bx-bx=a-ab^2$

Factoring, $b(a^2-1)x=a(1-b^2)$

Dividing by $b(a^2-1)$, $x=\dfrac{a(1-b^2)}{b(a^2-1)}$

20. Given $1.2x-\dfrac{.18x-.05}{.5}=.4x+8.9$

Multiplying by 50, $60x-18x+5=20x+445$

Transposing and uniting, $22x = 440$

Dividing by 22, $x = 20$

21. Given $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2$

Expanding, $ab + ax + bx + x^2 - ab - ac = \frac{a^2c}{b} + x^2$

Transposing, etc., $ax + bx = \frac{a^2c}{b} + ac$

Factoring, $(a+b)x = \frac{(a+b)ac}{b}$

Dividing by $a+b$, $x = \frac{ac}{b}$

22. Given $\frac{3x}{b} - \frac{x}{c} = m - c$

Multiplying by bc, $3cx - bx = bc(m-c)$

Dividing by $3c - b$, $x = \frac{bc(m-c)}{3c-b}$

23. Given $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$

Multiplying by 18, $8x + 6 + \frac{126x - 522}{5x - 12} = 8x + 19$

Transposing and uniting, $\frac{126x - 522}{5x - 12} = 13$

Clearing of fractions, $126x - 522 = 65x - 156$

Transposing and uniting, $61x = 366$

Dividing by 61, $x = 6$

24. Given $\frac{7x+9}{4} - \left(x - \frac{2x-1}{9}\right) = 7$

Mult. by 36, $63x + 81 - 36x + 8x - 4 = 252$

Transposing and uniting, $35x = 175$

Dividing $x = 5$

25. Given $\frac{(a-b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$

Mult. by 12, $6(a-b)x+4x=3ab+12a$

Dividing by $6(a-b)+4$, $x=\dfrac{3a(b+4)}{6(a-b)+4}$

26. Given $4.8x-\dfrac{.72x-.05}{.5}=1.6x+8.9$

Multiplying by 50, $240x-72x+5=80x+445$

Transposing and uniting, $88x=440$

Dividing by 88, $x=5$

27. Given $\dfrac{a-x}{b}-\dfrac{4a-x}{c}=a-b$

Multiplying by bc,

$$ac-cx-4ab+bx=abc-b^2c$$

Transposing, $bx-cx=4ab-ac+abc-b^2c$

Dividing by $b-c$, $x=\dfrac{4ab-ac+abc-b^2c}{b-c}$

28. Given $\dfrac{x}{a}-\dfrac{x}{b}+\dfrac{x}{c}=\dfrac{2}{ac}$

Multiplying by abc, $bcx-acx+abx=2b$

Dividing by $bc-ac+ab$, $x=\dfrac{2b}{bc-ac+ab}$

29. Given $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$

Factoring, $\dfrac{1}{a(b-x)}+\dfrac{1}{b(c-x)}=\dfrac{1}{a(c-x)}$

Multiplying by $ab(b-x)(c-x)$,

$$b(c-x)+a(b-x)=b(b-x)$$

Or, $bc-bx+ab-ax=b^2-bx$

Transposing, etc., $ax=ab-b^2+bc$

Whence, $x=\dfrac{b(a-b+c)}{a}$

30. Given $(x-\frac{5}{2})(x+\frac{3}{2})-(x-5)(x+3)-\frac{23}{4}=0$

Expanding, $x^2-x-\frac{15}{4}-x^2+2x+15-\frac{23}{4}=0$

Whence, $-x+2x+15-27=0$

Transposing and uniting, $x=12$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(ART. **160**, pp. 93–101.)

7. Let $x =$ value of chaise,

and $2x =$ value of horse.

Their sum, $x+2x=336$

Or, $3x=336$

Whence, $x=112$, value of chaise.

Then, $2x=224$, value of horse.

8. Let $x =$ no. oxen,

and $2x =$ no. cows.

Then $55x =$ value of oxen,

and $64x =$ value of cows.

Their sum, $119x=1428$

Whence, $x=12$, no. oxen.

Then, $2x=24$, no. cows.

9. Let $x =$ first number,

and $x+3 =$ second number.

By the conditions, $(x+3)^2-x^2=51$

Or, $x^2+6x+9-x^2=51$

Transposing and uniting, $6x=42$

Whence, $x=7$, first number.

Then, $x+3=10$, second number.

10. Let $x =$ each son's share,

$2x =$ each daughter's share,

and $6x =$ the wife's share.

Then, $4x =$ two daughters' share,

and $3x =$ three sons' share.

By the conditions, $6x + 4x + 3x = 1872$

Or, $13x = 1872$

Whence, $x = 144$, each son's share.

Then, $2x = 288$, each daughter's share.

Also, $6x = 864$, wife's share.

11. Let $x =$ no. days he labored,

and $36 - x =$ no. days he was absent.

Then $\frac{5x}{4} =$ sum rec'd for labor,

and $\frac{36 - x}{2} =$ sum forfeited by absence.

By the conditions, $\frac{5x}{4} - \frac{36 - x}{2} = 17$

Clearing of fract., $5x - 72 + 2x = 68$

Or, $7x = 140$

Whence, $x = 20$, days labored.

Then, $36 - x = 16$, days absent.

12. Let $x =$ the value of the saddle,

and $x + 114 =$ the value of the horse.

By the conditions, $\frac{2}{3}(x + 114) = 7x$

Clearing of fractions, $2x + 228 = 21x$

Or, $19x = 228$

Whence, $x = 12$, value of saddle.

Then, $x + 114 = 126$, value of horse.

13. Let $x =$ the number.

Then, $\frac{x - 7}{6} = 5$

Clearing of fractions, $x - 7 = 30$

Whence, $x = 37$, the number.

14. Let $x =$ no. artillery soldiers,

$2x =$ no. cavalry soldiers,

and $25x =$ no. infantry soldiers.

Their sum, $28x = 2744$

Whence, $x = 98$, no. artillery.

Then, $2x = 196$, no. cavalry.

Also, $25x = 2450$, no. infantry.

15. Let $x =$ side of the square,

and $x^2 =$ area of the square.

By the conditions, $(x+6)(x-4) = x^2$

Or, $x^2 + 2x - 24 = x^2$

Transposing, etc., $2x = 24$

Whence, $x = 12$

Then, $x^2 = 144$, the area.

16. Let $x =$ no. miles by water,

$\frac{4x}{7} =$ no. miles on foot,

and $\frac{2x}{5} =$ no. miles on horseback.

By the conditions, $x + \frac{4x}{7} + \frac{2x}{5} = 3036$

Clear. of fract., $35x + 20x + 14x = 106260$

Or, $69x = 106260$

Whence, $x = 1540$, no. m. by water.

Then, $\frac{4x}{7} = 880$, no. m. on foot.

Also, $\frac{2x}{5} = 616$, no. m. on horseback

17. Let $x =$ value of estate at first.

Then $\frac{5x}{4} - 760 =$ value after first change.

By the conditions, $\frac{5x}{4} - 760 + 600 = 2000$

Or, $\frac{5x}{4} = 2160$

Dividing by $\frac{5}{4}$, $x = 1728$, orig. value of est.

18. Let $x =$ whole sum.

Then $\frac{3x}{8} \times \frac{5}{100} =$ income at 5 per cent.,

and $\frac{5x}{8} \times \frac{6}{100} =$ income at 6 per cent.

The whole income, $\frac{3x}{160} + \frac{3x}{80} = 180$

Clearing of fractions, $3x + 6x = 28800$

Or, $9x = 28800$

Whence, $x = 3200$

Then, $\frac{3x}{8} = 1200$, sum at 5 per cent.

Also, $\frac{5x}{8} = 2000$, sum at 6 per cent.

19. Let $x =$ no. eggs at first.

Then $\frac{x-1}{2} =$ no. after first sale,

$\frac{x-1}{4} - \frac{1}{2} = \frac{x-3}{4} =$ no. after second sale,

and $\frac{x-3}{8} - \frac{1}{2} = \frac{x-7}{8} =$ no. after third sale.

By the conditions, $\frac{x-7}{8} = 0$

Or, $x - 7 = 0$

Whence $x = 7$, original no. of eggs.

20. Let $x =$ no. bushels at 9 s.,

and $40 - x =$ no. bushels at 13 s.

By the conditions, $9x + 13(40 - x) = 40 \times 10$

Expanding, $9x + 520 - 13x = 400$

Or, $4x = 120$

Whence, $x = 30$, no. bushels at 9 s.

Then, $40 - x = 10$, no. bushels at 13 s.

21. Let $x =$ the number.
Then, $3x + 12 - 54 = 144 - 3x$
Or, $6x = 186$
Whence, $x = 31$, the number.

22. Let $x =$ the sum B had.
Then, $\frac{5x - 60}{5} = \frac{x}{2}$
Clearing of fractions, $10x - 120 = 5x$
Or, $5x = 120$
Whence, $x = 24$, B's sum.

23. Let $x =$ no. dollars in the purse.
Then $\frac{x}{2} - 15 =$ A's share,
and $\frac{x}{4} + 13 =$ B's share.
By the conditions, $x = \frac{x}{2} - 15 + \frac{x}{4} + 13 + 27$
Or, $x = \frac{x}{2} + \frac{x}{4} + 25$
Clearing of fractions, $4x = 2x + x + 100$
Transposing and uniting, $x = 100$, whole amount.

24. Let $3x =$ A's distance,
and $5x =$ B's distance.
Then, $3x + 5x = 120$
Or, $8x = 120$
Whence, $x = 15$
Then, $3x = 45$, A's distance.
Also, $5x = 75$, B's distance.

25. Let $x =$ the first digit,
and $x - 4 =$ the second digit.
Then $10x + x - 4 =$ the number.
By the conditions, $\frac{11x - 4}{2x - 4} = 7$

Clearing of fractions, $11x-4=14x-28$
Transposing, uniting, etc., $3x=24$
Whence, $x=8$, first digit.
Then, $x-4=4$, second digit.
Also, $11x-4=84$, the number.

26. Let $x=$ no. acres bought.
Then $\frac{180}{x}=$ cost per acre,
and $\frac{180}{x-2}=$ am't rec'd per acre.

By the conditions, $\frac{180}{x}+\frac{180}{x}\times\frac{240}{100x}=\frac{180}{x-2}$
Or, $\frac{1}{x}+\frac{12}{5x^2}=\frac{1}{x-2}$
Clearing of fract., $5x^2-10x+12x-24=5x^2$
Transposing and uniting, $2x=24$
Whence, $x=12$, no. acres.

27 Let $7x=$ A's share,
and $8x=$ B's share.
Then, $15x=1000$
Whence, $x=66\frac{2}{3}$
Then, $7x=466\frac{2}{3}$, A's share,
Also, $8x=533\frac{1}{3}$, B's share.

28. Let $x=$ the sum let.
Then $\frac{21x}{20}=$ 1st amount,
$\frac{441x}{400}=$ 2d amount,
and $\frac{9261x}{8000}=$ 3d amount.

By the conditions, $\frac{9261x}{8000}=2315.25$

Clearing of fractions, $9261x = 18522000$

Whence, $x = 2000$, sum let.

29. Let $x =$ the cost of each.

Then $x - 1 = \frac{x + \frac{3}{4}}{2}$

Multiplying by 8, $8x - 8 = 4x + 3$

Transposing and uniting, $4x = 11$

Whence, $x = 2\frac{3}{4}$, cost of picture.

30. Let $x =$ no. days req. by B alone.

Then $\frac{1}{x} =$ part B does in a day.

By the conditions, $\frac{1}{x} + \frac{1}{10} = \frac{1}{7}$

Clearing of fractions, $70 + 7x = 10x$

Or, $3x = 70$

Whence, $x = 23\frac{1}{3}$, no. days req. by B.

31. Let $x =$ amount given each man,

and $8 - x =$ amount given each woman.

Then, $5x + 7(8 - x) = 46$

Or, $5x + 56 - 7x = 46$

Transposing, uniting, etc., $2x = 10$

Whence, $x = 5$

Then, $5x = 25$, am't rec'd by the men.

Also, $7(8 - x) = 21$, am't rec'd by the women.

32. Let $x =$ value of 1st farm.

Then $\frac{7}{9}(x + 183) =$ value of 2d farm.

By the conditions, $\frac{7x + 1281}{9} + 183 = \frac{13x}{8}$

Clear. of fract., $56x + 10248 + 13176 = 117x$

Transposing, uniting, etc., $61x = 23424$

Whence, $x = 384$, value of 1st farm.

Then, $\frac{7}{9}(x + 183) = 441$, value of 2d farm.

33. Let $x =$ no. hours after 6.
Then $x =$ no. hour spaces passed over by hour hand,
and $12x =$ no. hour spaces passed over by min. hand.
Therefore, $12x - x = 12$*
Or, $11x = 12$
Whence, $x = 1\frac{1}{11}$, no. hours after 6.
$1\frac{1}{11}$ hours $= 1$ h. $5\frac{5}{11}$ m.; 6 h. + 1 h. $5\frac{5}{11}$ m. = 7 h. $5\frac{5}{11}$ m.

34. Let $x =$ the time, when all are opened.
Then $\frac{1}{x} =$ part emptied by all in 1 m.
Therefore, $\frac{1}{80} + \frac{1}{200} + \frac{1}{300} = \frac{1}{x}$
Mult. by 1200, $15x + 6x + 4x = 1200$
Or, $25x = 1200$
Whence, $x = 48$, no. minutes required.

35. Let $9x =$ no. leaps by the fox,
and $6x =$ no. leaps by the greyhound.
Then $\frac{7}{3} \times 6x = 14x =$ no. fox's leaps in the whole distance passed over by the greyhound.
Therefore, $14x = 9x + 60$
Or, $5x = 60$
Whence, $x = 12$
Then, $9x = 108$, no. leaps by fox.
Also, $6x = 72$, no. leaps by greyhound.

36. Let one man be x times as strong as one boy.
Then, $7x + 3 = 9x - 9$
Or, $2x = 12$
Whence, $x = 6$

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* It is evident that the minute-hand must move once round the dial, or 12 hour spaces, more than the hour-hand, before they can again be opposite to each other. (See Key to Elementary Algebra, pp. 76, 77.)

37. Let $x =$ the first part,

and $43 - x =$ the second part.

By the conditions, $\frac{x - 20}{3} = 17 - (43 - x)$

Clearing of fractions, etc., $x - 20 = 51 - 129 + 3x$

Or, $2x = 58$

Whence, $x = 29$, first part.

Then, $43 - x = 14$, second part.

38 Let $x =$ the no. of yards start.

1 mile = 1760 yards; 2m. 54s. = 174s.

Then $\frac{1760}{176} = 10 =$ rate of loser per sec.

By the conditions, $x + 10 \times 174 = 1760$

Or, $x = 20$, no. yards start.

39. Let $x =$ no. oz. gold,

and $4160 - x =$ no. oz. silver.

Then $x \div 19\frac{1}{4} = \frac{4x}{77} = \left\{ \begin{array}{l} \text{no. oz. water displ.} \\ \text{by the gold,} \end{array} \right.$

and $(4160 - x) \div 10\frac{1}{2} = \frac{8320 - 2x}{21} = \left\{ \begin{array}{l} \text{no. oz. water displ.} \\ \text{by silver.} \end{array} \right.$

Hence, $\frac{4x}{77} + \frac{8320 - 2x}{21} = 250$

Clear. of fract., $12x + 91520 - 22x = 57750$

Or, $10x = 33770$

Whence, $x = 3377$, no. oz. gold.

Then, $4160 - x = 783$, no. oz. silver.

40. Let $x =$ no. hours A must work,

and $7 - x =$ no. hours B and C must work.

Then $\frac{1}{12} =$ part A does in 1 hour,

$\frac{1}{5} - \frac{1}{12} = \frac{7}{60} =$ part C does in 1 hour,

and $\frac{3}{2} \times \frac{7}{60} = \frac{7}{40} =$ part B does in 1 hour.

By the conditions, $\frac{x}{12} + (\frac{7}{60} + \frac{7}{40})(7 - x) = 1$

Or, $\frac{x}{12}+\frac{49-7x}{24}=1$

Clearing of fractions, $2x+49-7x=24$

Or, $5x=25$

Whence, $x=5$, A's time.

5 o'clock $+\,5$ h. $=10$ o'clock.

41. Let $x=$ his annual expenses.

Then $\frac{5}{4}(5120-x)=$ capital at end of 1st year,

$\frac{5}{4}\left(\frac{25600-9x}{4}\right)=$ capital at end of 2d year,

$\frac{5}{4}\left(\frac{128000-61x}{16}\right)=$ capital at end of 3d year,

and $\frac{5}{4}\left(\frac{640000-369x}{64}\right)=$ capital at end of 4th year.

Hence, $\frac{5}{4}\left(\frac{640000-369x}{64}\right)=3275$

Or, $\frac{640000-369x}{256}=655$

Clearing of fract., $640000-369x=167680$

Or, $369x=472320$

Whence, $x=1280$, annual expenses.

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

(Art. **169**, pp. 107 - 109.)

6. Given $\begin{cases}3x+7y=33 & (1)\\ 2x+4y=20 & (2)\end{cases}$

From (1), by transp. and divis., $x=\frac{33-7y}{3}$ (3)

From (2), by transp. and divis., $x=\frac{20-4y}{2}$ (4)

Combining (3) and (4), by Ax. 7, $\frac{20-4y}{2}=\frac{33-7y}{3}$ (5)

Clearing of fractions, $60-12y=66-14y$

Or, $2y=6$

Whence $y=3$

Substituting value of y in (4), $x=\frac{20-12}{2}=4$

7. Given $\left\{\begin{aligned} 7x+2y&=39 \quad (1)\\ 3x-4y&=7 \quad (2)\end{aligned}\right.$

Multiplying (1) by 2, $\underline{14x+4y=78}$ (3)

Adding (2) and (3), $17x=85$

Whence, $x=5$

Substituting value of x in (1), $35+2y=39$

Or, $2y=4$

Whence, $y=2$

8. Given $\left\{\begin{aligned} 6x-3y&=27 \quad (1)\\ \underline{4x-6y}&\underline{=-2} \quad (2)\end{aligned}\right.$

From (1), $y=2x-9$ (3)

Substituting value of y in (2), $4x-12x+54=-2$ (4)

Or, $8x=56$

Whence, $x=7$

Substituting value of x in (3), $y=14-9=5$

9. Given $\left\{\begin{aligned} 7x+3y&=62 \quad (1)\\ \underline{5x-2y}&\underline{=36} \quad (2)\end{aligned}\right.$

Multiplying (1) by 2, $14x+6y=124$ (3)

Multiplying (2) by 3, $\underline{15x-6y=108}$ (4)

Adding (3) and (4), $29x=232$

Whence $x=8$

Substituting in (1), $56+3y=62$

Or, $3y=6$

Whence, $y=2$

10. Given $\left\{\begin{aligned} 12x+8y&=116 \quad (1)\\ \underline{2x-y}&\underline{=3} \quad (2)\end{aligned}\right.$

From (2), $y=2x-3$ (3)

Substituting in (1), $12x+16x-24=116$ (4)

Or, $28x=140$

Whence, $x=5$

Substituting in (3), $y=10-3=7$

11. Given $\begin{cases} 11x+3y=124 & (1) \\ 2x-6y=-56 & (2) \end{cases}$

Equation (1), $11x+3y=124$
Dividing (2) by 2, $x-3y=-28$ (3)
Adding (1) and (3), $12x=96$
Whence, $x=8$
Substituting in (1), $88+3y=124$
Or, $3y=36$
Whence, $y=12$

12. Given $\begin{cases} 9x+4y=58 & (1) \\ 3x+2y=26 & (2) \end{cases}$

Multiplying (2) by 3, $9x+6y=78$ (3)
Equation (1), $9x+4y=58$
Subtracting (1) from (3), $2y=20$
Whence, $y=10$
Substituting in (2), $3x+20=26$
Or, $3x=6$
Whence, $x=2$

13. Given $\begin{cases} 6x+5y=112 & (1) \\ 8x-2y=80 & (2) \end{cases}$

From (2), $y=4x-40$ (3)
Substituting in (1), $6x+20x-200=112$ (4)
Or, $26x=312$
Whence, $x=12$
Substituting in (3), $y=48-40=8$

14. Given $\begin{cases} 7x-2y=-6 & (1) \\ 2x+2y=24 & (2) \end{cases}$

Adding (1) and (2), $9x=18$ (3)
Whence, $x=2$
Substituting in (2), $4+2y=24$

Or, $2y=20$

Whence, $y=10$

15. Given $\begin{cases} 6x+11y=115 & (1) \\ 8x-22y=-30 & (2) \end{cases}$

Dividing (2) by 2, $4x-11y=-15$ (3)

Equation (1), $6x+11y=115$

Adding (1) and (3), $10x=100$

Whence, $x=10$

Substituting in (1), $60+11y=115$

Or, $11y=55$

Whence, $y=5$

16. Given $\begin{cases} 2x+3y=47 & (1) \\ 10x-12y=-62 & (2) \end{cases}$

From (1), $y=\frac{47-2x}{3}$ (3)

From (2), $y=\frac{5x+31}{6}$ (4)

By Ax. 7, $\frac{5x+31}{6}=\frac{47-2x}{3}$ (5)

Clearing of fractions, $5x+31=94-4x$

Or, $9x=63$

Whence, $x=7$

Substituting in (3), $y=\frac{47-14}{3}=11$

17. Given $\begin{cases} \frac{x}{2}-\frac{y}{3}=0 & (1) \\ \frac{x}{4}+\frac{y}{6}=6 & (2) \end{cases}$

Clearing (1) of fractions, $3x-2y=0$ (3)

Clearing (2) of fractions, $3x+2y=72$ (4)

Adding (3) and (4), $6x=72$

Whence, $x=12$

Subtracting (3) from (4), $4y=72$

Whence, $y=18$

18. Given $\begin{cases} \frac{3x}{5}-y=11 & (1) \\ x+\frac{y}{5}=37 & (2) \end{cases}$

Clearing (1) of fractions, $3x-5y=55$ (3)

Clearing (2) of fractions, $5x+y=185$ (4)

From (4), $y=185-5x$ (5)

Substituting in (3), $3x-925+25x=55$ (6)

Or, $28x=980$

Whence, $x=35$

Substituting in (5), $y=185-175=10$

19. Given $\begin{cases} \frac{4x}{7}-\frac{2y}{3}=2 & (1) \\ x+7y=175 & (2) \end{cases}$

Clearing (1) of fractions, $12x-14y=42$ (3)

From (2), $x=175-7y$ (4)

Substituting in (3), $2100-84y-14y=42$ (5)

Or, $98y=2058$

Whence, $y=21$

Substituting in (4), $x=175-147=28$

20. Given $\begin{cases} \frac{7x}{8}-\frac{y}{9}=19 & (1) \\ 3x+3y=126 & (2) \end{cases}$

Clearing (1) of fractions, $63x-8y=1368$ (3)

Dividing (2) by 3, $x+y=42$ (4)

From (4), $y=42-x$ (5)

Substituting in (3), $63x-336+8x=1368$ (6)

Or, $71x=1704$

Whence, $x=24$

Substituting in (5), $y=42-24=18$

21. Given $\begin{cases} 14x + \frac{5y}{6} = 38 & (1) \\ x + 12y = 146 & (2) \end{cases}$

Clearing (1) of fractions, $84x + 5y = 228$ (3)

Multiplying (2) by 84, $84x + 1008y = 12264$ (4)

Subtracting (3) from (4), $1003y = 12036$

Whence, $y = 12$

Substituting in (2), $x + 144 = 146$

Whence, $x = 2$

22. Given $\begin{cases} \frac{x}{7} - \frac{7y}{10} = -20 & (1) \\ \frac{x}{4} + 3y = 134 & (2) \end{cases}$

Clearing (1) of fractions, $10x - 49y = -1400$ (3)

Clearing (2) of fractions, $x + 12y = 536$ (4)

From (4), $x = 536 - 12y$ (5)

Substituting in (3), $5360 - 120y - 49y = -1400$ (6)

Or, $169y = 6760$

Whence, $y = 40$

Substituting in (5), $x = 536 - 480 = 56$

23. Given $\begin{cases} ax + by = c & (1) \\ a'x + b'y = c' & (2) \end{cases}$

Multiplying (1) by a', $a'ax + a'by = a'c$ (3)

Multiplying (2) by a, $a'ax + ab'y = ac'$ (4)

Subtracting (3) from (4), $(ab' - a'b)y = ac' - a'c$ (5)

Whence, $y = \frac{ac' - a'c}{ab' - a'b}$ (6)

Multiplying (1) by b', $ab'x + b'by = b'c$ (7)

Multiplying (2) by b, $a'bx + b'by = bc'$ (8)

Subtracting (8) from (7), $(ab' - a'b)x = b'c - bc'$

Whence, $x = \frac{b'c - bc'}{ab' - a'b}$

24. Given $\begin{cases} \frac{x}{a} - \frac{y}{b} = m & (1) \\ \frac{x}{c} + \frac{y}{d} = n & (2) \end{cases}$

Clearing (1) of fractions, $bx - ay = abm$ (3)

Clearing (2) of fractions, $dx + cy = cdn$ (4)

Multiplying (3) by c, $bcx - acy = abcm$ (5)

Multiplying (4) by a, $adx + acy = acdn$ (6)

Adding (5) and (6), $(ad + bc)x = abcm + acdn$ (7)

Whence, $x = \frac{abcm + acdn}{ad + bc}$ (8)

Multiplying (3) by d, $bdx - ady = abdm$ (9)

Multiplying (4) by b, $bdx + bcy = bcdn$ (10)

Subtracting (9) from (10), $(ad + bc)y = bcdn - abdm$

Whence, $y = \frac{bcdn - abdm}{ad + bc}$

25. Given $\begin{cases} \frac{x}{2} - 12 = \frac{y}{4} + 8 & (1) \\ \frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27 & (2) \end{cases}$

Transposing and uniting in (1), $\frac{x}{2} - \frac{y}{4} = 20$ (3)

Clearing (3) of fractions, $2x - y = 80$ (4)

From (2), $\frac{x+y}{5} + \frac{3}{x} - \frac{2y - x}{4} = 35$ (5)

Clearing (5) of fractions,

$12x + 12y + 20x - 30y + 15x = 2100$ (6)

Uniting terms, $47x - 18y = 2100$ (7)

From (4), $y = 2x - 80$ (8)

Substituting in (7), $47x - 36x + 1440 = 2100$ (9)

Or, $11x = 660$

Whence, $x = 60$

Substituting in (8), $y = 120 - 80 = 40$

SIMPLE EQUATIONS

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(Art. **170**, pp. 109–112.)

3. Given $$\left\{\begin{array}{l} 3x-y-2z=0 \qquad (1)\\ 6x+2y+3z=45 \qquad (2)\\ 4x+3y-z=31 \qquad (3)\end{array}\right.$$

Multiplying (1) by (3), $\underline{9x-3y-6z=0}$ (4)

Adding (3) and (4), $13x-7z=31$ (5)

Multiplying (1) by 2, $6x-2y-4z=0$ (6)

Equation (2), $\underline{6x+2y+3z=45}$

Adding (2) and (6), $12x-z=45$ (7)

Multiplying (7) by 7, $84x-7z=315$ (8)

Equation (5), $\underline{13x-7z=31}$

Subtracting (5) from (8), $71x=284$

Whence $x=4$

Substituting in (7), $48-z=45$

Whence, $z=3$

Substituting in (1), $12-y-6=0$

Whence, $y=6$

4. Given $$\left\{\begin{array}{l} 8x-9y-7z=-36 \qquad (1)\\ 12x-y-3z=36 \qquad (2)\\ \underline{6x-2y-z=10} \qquad (3)\end{array}\right.$$

From (3), $z=6x-2y-10$ (4)

Substituting in (1),

$$8x-9y-42x+14y+70=-36 \qquad (5)$$

Substituting in (2),

$$12x-y-18x+6y+30=36 \qquad (6)$$

Reducing (5), $34x-5y=106$ (7)

Reducing (6), $\underline{-6x+5y=6}$ (8)

Adding (7) and (8), $28x=112$

Whence, $x = 4$

Substituting in (8), $5y - 24 = 6$

Or, $5y = 30$

Whence, $y = 6$

Substituting in (4), $z = 24 - 12 - 10 = 2$

5. Given
$$\begin{cases} 7x + 4y - z = 78 & (1) \\ 4x - 5y - 3z = -21 & (2) \\ x - 3y - 4z = -37 & (3) \end{cases}$$

Multiplying (1) by 4, $28x + 16y - 4z = 312$ (4)

Subtracting (3) from (4), $27x + 19y = 349$ (5)

Multiplying (1) by 3, $21x + 12y - 3z = 234$ (6)

Equation (2), $4x - 5y - 3z = -21$

Subtracting (2) from (6), $17x + 17y = 255$ (7)

Dividing (7) by 17, $x + y = 15$ (8)

Multiplying (8) by 19, $19x + 19y = 285$ (9)

Equation (5), $27x + 19y = 349$

Subtracting (9) from (5), $8x = 64$ (10)

Whence, $x = 8$

Substituting in (8), $8 + y = 15$

Whence, $y = 7$

Substituting in (1), $56 + 28 - z = 78$

Whence, $z = 6$

7. Given
$$\begin{cases} x + y = 30 & (1) \\ x + z = 25 & (2) \\ y + z = 15 & (3) \end{cases}$$

Subtracting (3) from (2), $x - y = 10$ (4)

Equation (1), $x + y = 30$

Adding (1) and (4), $2x = 40$

Whence, $x = 20$

Subtracting (4) from (1), $2y = 20$

Whence, $y = 10$

Substituting in (3), $10 + z = 15$

Whence, $z = 5$

Note. The above equations may also be solved by the use of s, as in the sixth example.

8. Given
$$\begin{cases} 8x - 4y = 24 - z & (1) \\ 6x + y = z + 84 & (2) \\ x + 80 = 3y + 4z & (3) \end{cases}$$

Adding (1) and (2), $14x - 3y = 108$ (4)

Multiplying (1) by 4, $32x - 16y = 96 - 4z$ (5)

Equation (3), $x - 3y = -80 + 4z$

Adding (5) and (3), $33x - 19y = 16$ (6)

From (6), $y = \dfrac{33x - 16}{19}$ (7)

From (4), $y = \dfrac{14x - 108}{3}$ (8)

By Ax. 7, $\dfrac{33x - 16}{19} = \dfrac{14x - 108}{3}$ (9)

Clearing of fractions, $99x - 48 = 266x - 2052$ (10)

Or, $167x = 2004$

Whence, $x = 12$

Substituting in (8), $y = \dfrac{168 - 108}{3} = 20$

Substituting in (1), $96 - 80 = 24 - z$

Whence, $z = 8$

9. Given
$$\begin{cases} \frac{x}{2} + \frac{y}{3} - \frac{z}{4} = 23 & (1) \\ \frac{x}{3} - \frac{y}{4} + \frac{z}{2} = 12 & (2) \\ \frac{x}{4} + \frac{y}{2} - \frac{z}{3} = 17 & (3) \end{cases}$$

Multiplying (1) by 48, $24x + 16y - 12z = 1104$ (4)

Multiplying (2) by 24, $8x - 6y + 12z = 288$ (5)

Multiplying (3) by 36, $\underline{9x + 18y - 12z = 612}$ (6)

Adding (5) and (6), $17x + 12y = 900$ (7)

Adding (4) and (5), $32x + 10y = 1392$ (8)

Multiplying (7) by 5, $85x + 60y = 4500$ (9)

Multiplying (8) by 6, $\underline{192x + 60y = 8352}$ (10)

Subtracting (9) from (10), $107x = 3852$

Whence, $x = 36$

Substituting in (7), $612 + 12y = 900$

Or, $12y = 288$

Whence, $y = 24$

Substituting in (5), $288 - 144 + 12z = 288$

Or, $12z = 144$

Whence, $z = 12$

10. Given

$$\begin{cases} 3u + x + 2y - z = 22 & (1) \\ 4x - y + 3z = 35 & (2) \\ 4u + 3x - 2y = 19 & (3) \\ 2u + 4y + 2z = 46 & (4) \end{cases}$$

Multiplying (1) by 3, $9u + 3x + 6y - 3z = 66$ (5)

Equation (2), $\underline{4x - y + 3z = 35}$

Adding (5) and (2), $9u + 7x + 5y = 101$ (6)

Equation (1), $3u + x + 2y - z = 22$

Dividing (4) by 2, $\underline{u + 2y + z = 23}$ (7)

Adding (1) and (7), $4u + x + 4y = 45$ (8)

Multiplying (3) by 2, $\underline{8u + 6x - 4y = 38}$ (9)

Adding (8) and (9), $12u + 7x = 83$ (10)

Multiplying (3) by 5, $20u + 15x - 10y = 95$ (11)

Multiplying (6) by 2, $\underline{18u + 14x + 10y = 202}$ (12)

Adding (11) and (12), $38u + 29x = 297$ (13)

Multiplying (13) by 6, $228u + 174x = 1782$ (14)

Multiplying (10) by 19, $\underline{228u + 133x = 1577}$ (15)

Subtracting (15) from (14), $41x = 205$

Whence, $x = 5$

Substituting in (10), $12u + 35 = 83$

Or, $12u = 48$

Whence, $u = 4$

Substituting in (3), $16 + 15 - 2y = 19$

Or, $2y = 12$

Whence, $y = 6$

Substituting in (7), $4 + 12 + z = 23$

Whence, $z = 7$

11. Given
$$\begin{cases} a = y + z & (1) \\ b = x + z & (2) \\ \underline{c = x + y} & (3) \end{cases}$$

Subtracting (1) from (2), $x - y = b - a$ (4)

Equation (3), $\underline{x + y = c}$

Adding (3) and (4), $2x = b + c - a$

Whence, $x = \frac{1}{2}(b + c - a)$

Subtracting (4) from (3), $2y = c - b + a$

Whence, $y = \frac{1}{2}(a + c - b)$

Substituting in (1), $a = \frac{1}{2}(a + c - b) + z$

Whence, $z = \frac{1}{2}(a + b - c)$

Note. The above equations may also be solved by subtracting each equation from half the sum of the three.

12. Given
$$\begin{cases} 4x - 4y = a + 4z & (1) \\ 6y - 2x = a + 2z & (2) \\ \underline{7z - y = a + x} & (3) \end{cases}$$

Dividing (2) by 2, $-x + 3y - z = \frac{1}{2}a$ (4)

Dividing (1) by 4, $x - y - z = \frac{1}{4}a$ (5)

Equation (3), $\underline{-x - y + 7z = a}$

Adding (4) and (5), $2y - 2z = \frac{3}{4}a$ (6)

Adding (3) and (5) $-2y + 6z = \frac{5}{4}a$ (7)

Adding (6) and (7), $4z = 2a$

Whence, $z = \frac{1}{2}a$

Substituting in (6), $2y - a = \frac{3}{4}a$

Or, $2y = \frac{7}{4}a$

Whence, $y = \frac{7}{8}a$

Substituting in (5), $x - \frac{7}{8}a - \frac{1}{2}a = \frac{1}{4}a$

Whence, $x = \frac{13}{8}a$

13. Given
$$\begin{cases} 2x + 2y + z = 35 - 2u & (1) \\ y + 3z = 31 & (2) \\ 4x + z = 32 & (3) \\ \frac{1}{3}x + 3y = 23 & (4) \end{cases}$$

Multiplying (3) by 3, $12x + 3z = 96$ (5)

Equation (2), $y + 3z = 31$

Subtracting (2) from (5), $12x - y = 65$ (6)

Multiplying (6) by 9, $108x - 9y = 585$ (7)

Multiplying (4) by 3, $x + 9y = 69$ (8)

Adding (7) and (8), $109x = 654$

Whence, $x = 6$

Substituting in (6), $72 - y = 65$

Whence, $y = 7$

Substituting in (3), $24 + z = 32$

Whence, $z = 8$

Substituting in (1), $12 + 14 + 8 = 35 - 2u$

Or, $2u = 1$

Whence, $u = \frac{1}{2}$

PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

(Pp. 112–119.)

4. Let $x =$ A's age,

and $y =$ B's age.

By the conditions, $\frac{x}{5}+\frac{2y}{3}=19\frac{1}{3}$ (1)

and $\frac{7x}{8}-\frac{2y}{5}=18\frac{1}{4}$ (2)

Clearing (1) of fractions, $3x+10y=290$ (3)

Clearing (2) of fractions, $35x-16y=730$ (4)

Multiplying (3) by 8, $24x+80y=2320$ (5)

Multiplying (4) by 5, $175x-80y=3650$ (6)

Adding (5) and (6), $199x=5970$

Whence, $x=30$, A's age.

Substituting in (3), $90+10y=290$

Or, $10y=200$

Whence, $y=20$, B's age.

5. Let $x=$ the numerator,

and $y=$ the denominator.

Therefore, $\frac{x}{y}=$ the fraction.

Then, $\frac{x+1}{y}=\frac{1}{3}$ (1)

and $\frac{x}{y+1}=\frac{1}{4}$ (2)

From (1), $3x-y=-3$ (3)

From (2), $4x-y=1$ (4)

Subtracting (3) from (4), $x=4$

Substituting in (3), $12-y=-3$

Whence, $y=15$

Therefore, $\frac{x}{y}=\frac{4}{15}$, the fraction.

6. Let $x=$ no. oxen,

and $y=$ no. cows.

Then, $x+y=89$ (1)

and $x-4-(y-20)=7$ (2)

From (2), $x-y=-9$ (3)
Equation (1), $x+y=89$
Adding (3) and (1), $2x=80$
Whence, $x=40$, no. oxen.
Subtracting (3) from (1), $2y=98$
Whence, $y=49$, no. cows.

7. Let $x=$ A's property,
and $y=$ B's property.

Then, $7x+\frac{y}{7}=990$ (1)

and $\frac{x}{7}+7y=510$ (2)

Multiplying (1) by 7, $49x+y=6930$ (3)
Multiplying (2) by 343, $49x+2401y=174930$ (4)
Subtracting (3) from (4), $2400y=168000$ (5)
Whence, $y=70$, B's property.
Substituting in (3), $49x+70=6930$
Or, $49x=6860$
Whence, $x=140$, A's property.

8. Let $x=$ A's age,
and $y=$ B's age.

Then, $y-\frac{x}{7}+5=6$ (1)

and $\frac{y}{5}+4=\frac{x}{14}$ (2)

From (1), $-x+7y=7$ (3)
From (2), $5x-14y=280$ (4)
Multiplying (3) by 2, $-2x+14y=14$ (5)
Adding (4) and (5), $3x=294$
Whence, $x=98$, A's age.
Substituting in (1), $y-14=1$
Whence, $y=15$, B's age.

9. Let $x =$ the larger part,
and $y =$ the smaller part.
Then, $x + y = 50$ (1)
and $\frac{3x}{8} = \frac{2y}{3}$ (2)
From (2), $9x = 16y$ (3)
Or, $x = \frac{16y}{9}$ (4)
Substituting in (1), $\frac{16y}{9} + y = 50$ (5)
Clearing of fractions, $16y + 9y = 450$ (6)
Or, $25y = 450$
Whence, $y = 18$, smaller part.
Substituting in (4), $x = \frac{16 \times 18}{9} = 32$, larger part.

10. Let $x =$ the man's age,
and $y =$ the wife's age.
Then $y : x :: 3 : 4$ (1)
and $y + 12 : x + 12 :: 5 : 6$ (2)
From (1), $3x = 4y$ (3)
From (2), $5x + 60 = 6y + 72$ (4)
Or, $5x - 6y = 12$ (5)
From (3), $y = \frac{3x}{4}$ (6)
Substituting in (5), $5x - \frac{9x}{2} = 12$
Clearing of fractions, $10x - 9x = 24$
Whence, $x = 24$, man's age.
Substituting in (6), $y = \frac{3 \times 24}{4} = 18$, wife's age.

11. Let $x =$ no. days he labored,
and $y =$ no. days he was absent.

Then, $x+y=10$ (1)
and $12x-8y=40$ (2)
Dividing (2) by 4, $3x-2y=10$ (3)
Multiplying (1) by 2, $2x+2y=20$ (4)
Adding (3) and (4), $5x=30$
Whence, $x=6$, days labored.
Substituting in (1), $6+y=10$
Whence, $y=4$, days absent.

12. Let $x=$ price of chaise,
and $y=$ price of horse.
Then, $x+y=208$ (1)
and $\frac{4x}{7}=\frac{2y}{3}$ (2)
Clearing (2) of fractions, $12x=14y$ (3)
Whence, $x=\frac{7y}{6}$ (4)
Substituting in (1), $\frac{7y}{6}+y=208$ (5)
Clearing of fractions, $7y+6y=1248$
Or, $13y=1248$
Whence, $y=96$, price of horse.
Substituting in (4), $x=\frac{7\times 96}{6}=112$, price of chaise.

13. Let $x=$ A's loss,
and $y=$ B's loss.
Then, $x=2y$ (1)
and $240-x=3(96-y)$ (2)
From (2), $3y-x=48$ (3)
Substituting $2y$ for x, $3y-2y=48$ (4)
Or, $y=48$, B's loss.
Substituting in (1), $x=2\times 48=96$, A's loss.

14. Let $x =$ time required by A,

and $y =$ time required by B.

Then, $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$ (1)

and $\frac{4}{x} + \frac{20}{y} = 1$ (2)

Dividing (2) by 4, $\frac{1}{x} + \frac{5}{y} = \frac{1}{4}$ (3)

Multiplying (1) by 5, $\frac{5}{x} + \frac{5}{y} = \frac{1}{2}$ (4)

Subtracting (3) from (4), $\frac{4}{x} = \frac{1}{4}$.

Whence, $x = 16$, A's time.

15. Let $x =$ no. bush. at 60 c.,

and $y =$ no. bush. at 90 c.

Then, $x + y = 40$ (1)

and $60x + 90y = 40 \times 80$ (2)

Dividing (2) by 10, $6x + 9y = 320$ (3)

Multiplying (1) by 6, $6x + 6y = 240$ (4)

Subtracting (4) from (3), $3y = 80$

Whence, $y = 26\frac{2}{3}$, bush. at 90 c.

Substituting in (1), $x + 26\frac{2}{3} = 40$

Whence, $x = 13\frac{1}{3}$, bush. at 60 c.

16. Let $x =$ no. bushels wheat,

and $y =$ no. bushels rye.

Then, $7x - 4y = 3$ (1)

and $x : y :: 3 : 5$ (2)

From (2), $y = \frac{5x}{3}$ (3)

Substituting in (1), $7x - \frac{20x}{3} = 3$ (4)

Clearing of fractions, $21x - 20x = 9$

Or, $x = 9$, bushels wheat.

Substituting in (3), $y = \frac{5 \times 9}{3} = 15$, bushels rye.

17. Let $x =$ income tax,

and $y =$ assessed tax.

Then, $x + y = 50$ (1)

and $\frac{3x}{2} + \frac{3y}{4} = 52\frac{1}{2}$ (2)

Multiplying (2) by $\frac{4}{3}$, $2x + y = 70$ (3)

Equation (1), $x + y = 50$

Subtracting (1) from (3), $x = 20$, income tax.

Substituting in (1), $20 + y = 50$

Whence, $y = 30$, assessed tax.

18. Let $x =$ A's original stock,

and $y =$ B's original stock.

Then $\frac{5y}{6} =$ A's accumulated stock,

$\frac{5y}{6} - x =$ A's profit,

and $\frac{x + 200}{6} =$ B's profit.

By the conditions, $\frac{5y}{6} - x + \frac{x + 200}{6} = 200$ (1)

and $x : y :: \frac{5y}{6} - x : \frac{x + 200}{6}$ (2)

Clearing (1) of fract., $5y - 6x + x + 200 = 1200$ (3)

Or, $5y - 5x = 1000$ (4)

Dividing by 5, $y - x = 200$ (5)

From (2), $\frac{x^2 + 200x}{6} = \frac{5y^2}{6} - xy$ (6)

Clearing of fractions, $x^2 + 200x = 5y^2 - 6xy$ (7)

From (5), $x + 200 = y$ (8)

Dividing (7) by (8), $x = 5y - 6x$ (9)

Whence, $x = \frac{5y}{7}$ (10)

Substituting in (5), $y - \frac{5y}{7} = 200$

Clearing of fractions, $7y - 5y = 1400$

Or, $2y = 1400$

Whence, $y = 700$, B's stock.

Substituting in (10), $x = \frac{5 \times 700}{7} = 500$, A's stock.

19. Let $x =$ no. cents,

and $y =$ no. oranges.

Then $\frac{x}{y} =$ price per orange.

By the conditions, $\frac{x}{y+5} = \frac{x}{y} - \frac{1}{2}$ (1)

and $\frac{x}{y-3} = \frac{x}{y} + \frac{1}{2}$ (2)

Clearing (1) of fractions, $2xy = 2xy + 10x - y^2 - 5y$ (3)

Clearing (2) of fractions, $2xy = 2xy - 6x + y^2 - 3y$ (4)

From (3), $10x = y^2 + 5y$ (5)

From (4), $6x = y^2 - 3y$ (6)

Subtracting (6) from (5), $4x = 8y$ (7)

Whence, $x = 2y$ (8)

Substituting in (6), $12y = y^2 - 3y$

Or, $y^2 = 15y$

Dividing by y, $y = 15$, no. oranges.

Substituting in (8), $x = 2 \times 15 = 30$, no. cents.

20. Let $x =$ price of first,

$y =$ price of second,

and $z =$ price of third.

Then, $3x + 4y + 2z = 60$ (1)

$4x + y + 5z = 59$ (2)

and $x + 10y + 3z = 90$ (3)

Multiplying (3) by 3, $3x + 30y + 9z = 270$ (4)

Equation (1), $3x + 4y + 2z = 60$

Subtracting (1) from (4), $26y + 7z = 210$ (5)

Multiplying (3) by 4, $4x + 40y + 12z = 360$ (6)

Equation (2), $4x + y + 5z = 59$

Subtracting (2) from (6), $39y + 7z = 301$ (7)

Subtracting (5) from (7), $13y = 91$

Whence, $y = 7$, price 2d.

Substituting in (5), $182 + 7z = 210$

Or, $7z = 28$

Whence, $z = 4$, price 3d.

Substituting in (3), $x + 70 + 12 = 90$

Or, $x = 8$, price 1st.

21. Let $x =$ value of harness,

$y =$ value of better horse,

and $z =$ value of poorer horse.

Then, $x + y + z = 120$ (1)

$x + z = 2y$ (2)

and $x + y = 3z$ (3)

Subtracting (3) from (1), $z = 120 - 3z$

Or, $4z = 120$

Whence, $z = 30$, value of poorer horse.

Subtracting (2) from (1), $y = 120 - 2y$

Or, $3y = 120$

Whence, $y = 40$, value of better horse.

Substituting in (1), $x + 40 + 30 = 120$

Whence, $x = 50$, value of harness.

22. Let x, y, and z represent the numbers.

Then, $x+\frac{y+z}{2}=34$ (1)

$y+\frac{x+z}{3}=34$ (2)

and $z+\frac{x+y}{4}=34$ (3)

Clearing (1) of fractions, $2x+y+z=68$ (4)

Clearing (2) of fractions, $x+3y+z=102$ (5)

Clearing (3) of fractions, $x+y+4z=136$ (6)

Subtracting (4) from (5), $-x+2y=34$ (7)

Multiplying (5) by 4, $4x+12y+4z=408$ (8)

Equation (6), $x+y+4z=136$

Subtracting (6) from (8), $3x+11y=272$ (9)

Multiplying (7) by 3, $-3x+6y=102$ (10)

Adding (9) and (10), $17y=374$

Whence, $y=22$, 2d number.

Substituting in (9), $3x+242=272$

Or, $3x=30$

Whence, $x=10$, 1st number.

Substituting in (3), $z+8=34$

Whence, $z=26$, 3d number.

23. Let $x-y=$ digit in hundreds' place,

$x=$ digit in tens' place,

and $x+y=$ digit in units' place.

Then $100(x-y)+10x+x+y=111x-99y=$ the number,

and $100(x+y)+10x+x-y=111x+99y=$ the number, with digits inverted.

By the conditions, $\frac{111x-99y}{\frac{3x}{2}}=41$ (1)

and $111x-99y+396=111x+99y$ (2)

Transposing and uniting in (2), $198y = 396$

Whence, $y = 2$

Clearing (1) of fractions, $74x - 66y = 41x$

Or, $33x = 66y$

Whence, $x = 2y = 4$

Therefore, $111x - 99y = 246$, the number.

24. Let $x =$ A's property,

$y =$ B's property,

$z =$ C's property,

and $w =$ D's property.

Then, $x + y + z + w = 14000 \quad (1)$

$2x + 3y + \frac{z}{2} + \frac{w}{5} = 16000 \quad (2)$

$x + 2y + 2z + \frac{2w}{5} = 18000 \quad (3)$

and $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} + \frac{w}{5} = 4000 \quad (4)$

Subtracting (3) from 2 times (1), $x + \frac{8w}{5} = 10000 \quad (5)$

Clearing (4) of fractions, $30x + 20y + 15z + 12w = 240000 \quad (6)$

Multiplying (1) by 20, $20x + 20y + 20z + 20w = 280000 \quad (7)$

Subtracting (6) from (7), $-10x + 5z + 8w = 40000 \quad (8)$

Clearing (2) of fractions, $20x + 30y + 5z + 2w = 160000 \quad (9)$

Multiplying (1) by 30, $30x + 30y + 30z + 30w = 420000 \quad (10)$

Subtracting (9) from (10), $10x + 25z + 28w = 260000 \quad (11)$

Multiplying (8) by 5, $-50x + 25z + 40w = 200000 \quad (12)$

Subtracting (12) from (11), $60x - 12w = 60000 \quad (13)$

Dividing (13) by 12, $5x - w = 5000 \quad (14)$

Clearing (5) of fractions, $5x + 8w = 50000 \quad (15)$

Subtracting (14) from (15), $9w = 45000$

Whence, $w = 5000$, D's prop.

Substituting in (5), $x + 8000 = 10000$

Whence, $x = 2000$, A's prop.

Substituting in (8), $-20000 + 5z + 40000 = 40000$

Or, $5z = 20000$

Whence, $z = 4000$, C's prop.

Substituting in (1), $2000 + y + 4000 + 5000 = 14000$

Whence, $y = 3000$, B's prop.

25. Let $x =$ no. A's turkeys,

and $y =$ no. B's turkeys.

Then, $x + 5 = y - 5$ (1)

and $\frac{3}{7}(y + 15) = x - 15$ (2)

Clearing (2) of fractions, $3y + 45 = 7x - 105$ (3)

Or, $7x - 3y = 150$ (4)

Multiplying (1) by 3, $3x - 3y = -30$ (5)

Subtracting (5) from (4), $4x = 180$

Whence, $x = 45$, A's number.

Substituting in (1), $y - 5 = 45 + 5$

Whence, $y = 55$, B's number.

26. Let $\frac{x}{100} =$ 1st rate,

and $\frac{y}{100} =$ 2d rate.

Then, $\frac{30000x}{100} - \frac{20000y}{100} = 800$ (1)

and $\frac{35000y}{100} - \frac{24000x}{100} = 310$ (2)

From (1), $3x - 2y = 8$ (3)

From (2), $-24x + 35y = 31$ (4)

Multiplying (3) by 8, $24x - 16y = 64$ (5)

Adding (4) and (5), $19y = 95$

Whence, $y=5$

Substituting in (3), $3x-10=8$

Or, $3x=18$

Whence, $x=6$

Hence the first rate is $\frac{6}{100}$, or 6 per cent., and the second rate is $\frac{5}{100}$, or 5 per cent.

27. Let $x=$ A's money,

$y=$ B's money,

and $z=$ C's money.

Then, $x+\frac{y+z}{2}=55$ (1)

$y+\frac{x+z}{3}=50$ (2)

and $z+\frac{x+y}{5}=50$ (3)

Clearing (1) of fractions, $2x+y+z=110$ (4)

Clearing (2) of fractions, $x+3y+z=150$ (5)

Clearing (3) of fractions, $x+y+5z=250$ (6)

Subtracting (5) from (6), $-2y+4z=100$ (7)

Multiplying (6) by 2, $2x+2y+10z=500$ (8)

Subtracting (4) from (8), $y+9z=390$ (9)

Dividing (7) by 2, $-y+2z=50$ 10)

Adding (9) and (10), $11z=440$

Whence, $z=40$, C's.

Substituting in (9), $y+360=390$

Whence, $y=30$, B's.

Substituting in (5), $x+90+40=150$

Whence, $x=20$, A's.

28. Let $x=$ oldest son's share,

$y=$ second son's share,

$z=$ third son's share,

and $w=$ youngest son's share.

Then, $\frac{y+z+w}{2} = x$ (1)

$\frac{x+z+w}{3} = y$ (2)

$\frac{x+y+w}{4} = z$ (3)

and $x - w = 14$ (4)

From (1), $2x - y - z - w = 0$ (5)

From (2), $x - 3y + z + w = 0$ (6)

From (3), $x + y - 4z + w = 0$ (7)

Adding (5) and (6), $3x - 4y = 0$ (8)

Whence, $y = \frac{3x}{4}$ (9)

Subst. in (5) and reducing, $5x - 4z - 4w = 0$ (10)

Subst. in (7) and reducing, $7x - 16z + 4w = 0$ (11)

Multiplying (10) by 4, $20x - 16z - 16w = 0$ (12)

Subtracting (11) from (12), $13x - 20w = 0$ (13)

Multiplying (4) by 20, $20x - 20w = 280$ (14)

Subtracting (13) from (14), $7x = 280$

Whence, $x = 40$, 1st share.

Substituting in (9), $y = \frac{3 \times 40}{4} = 30$, 2d share.

Substituting in (4), $40 - w = 14$

Whence, $w = 26$, 4th share.

Substituting in (10), $200 - 4z - 104 = 0$

Or, $4z = 96$

Whence, $z = 24$, 3d share.

Therefore, $x + y + z + w = 120$, whole sum.

GENERAL SOLUTION OF PROBLEMS.

(Art. 172, pp. 119–123.)

3. Let x represent the number of hours required. Then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, will represent the fractional part of the whole cistern filled by each pipe, respectively, in 1 hour, and $\frac{1}{x}$ the part filled by all the pipes running together.

Therefore, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$$

Clearing of fract., $bcx+acx+abx=abc$

Factoring, etc., $(ab+ac+bc)x=abc$

Whence, $$x=\frac{abc}{ab+ac+bc}, \text{ no. hours.}$$

4. $$\frac{abc}{ab+ac+bc}=\frac{2\times 5\times 10}{10+20+50}=\frac{100}{80}=1\tfrac{1}{4}, \text{ no. hours.}$$

6. $$\frac{ma}{m+n+p}=\frac{300\times 320}{300+500+800}=\frac{96000}{1600}=60, \text{ A's share.}$$

$$\frac{na}{m+n+p}=\frac{500\times 320}{300+500+800}=\frac{160000}{1600}=100, \text{ B's share.}$$

$$\frac{pa}{m+n+p}=\frac{800\times 320}{300+500+800}=\frac{256000}{1600}=160, \text{ C's share.}$$

7. Let mx and nx represent the parts.

Then, $$mx+nx=a$$

Whence, $$x=\frac{a}{m+n}$$

Therefore, $$mx=\frac{ma}{m+n}, \text{ 1st part.}$$

Also, $$nx=\frac{na}{m+n}, \text{ 2d part.}$$

8. Let $x=$ no. days second travels.

and $x+n=$ no. days first travels.

Therefore, $bx = a(x + n)$

Or, $bx = ax + na$

Transposing, $bx - ax = na$

Whence, $x = \frac{na}{b-a}$, no. days required.

9. Let x = the principal.

Then, $x + \frac{rtx}{100} = a$

Clearing of fractions, $100x + rtx = 100a$

Whence, $x = \frac{100a}{100 + rt}$, the principal.

10. Let x = the number of beggars.

Then, $ax - b = cx + d$

Or, $ax - cx = b + d$

Whence, $x = \frac{b+d}{a-c}$

11. Let x = number of years.

Then, $p + \frac{rpx}{100} = a$

Or, $\frac{rpx}{100} = a - p$

Whence, $x = \frac{100(a-p)}{rp}$

12. Let x = no. pieces 1st kind,

and $c - x$ = no. pieces 2d kind.

Then, $\frac{x}{a} + \frac{c-x}{b} = 1$

Clearing of fractions, $bx + ac - ax = ab$

Or, $bx - ax = ab - ac$

Whence, $x = \frac{a(b-c)}{b-a} = \frac{a(c-b)}{a-b}$, no. 1st kind.

Therefore, $c - x = c - \frac{ac - ab}{a-b} = \frac{b(a-c)}{a-b}$, no. 2d kind.

13. $\frac{a(c-b)}{a-b}=\frac{10\times(-5)}{-10}=5$, no. 1st kind.

$\frac{b(a-c)}{a-b}=\frac{20\times(-5)}{-10}=10$, no. 2d kind.

14. Let $x=$ cost of mixture per lb.

Then, $(a+b+c)x=ma+nb+pc$

Whence, $x=\frac{ma+nb+pc}{a+b+c}$, cost.

15. Let $mtx=$ A's share,

$nt'x=$ B's share,

and $pt''x=$ C's share.

Then, $mtx+nt'x+pt''x=a$

Whence, $x=\frac{a}{mt+nt'+pt''}$

Therefore, $mtx=\frac{mta}{mt+nt'+pt''}$

And, $nt'x=\frac{nt'a}{mt+nt'+pt''}$

Also, $pt''x=\frac{pt''a}{mt+nt'+pt''}$

INTERPRETATION OF NEGATIVE RESULTS.

(Art. 181, pp. 126–128.)

3. Let $x=$ the quantity to be added.

Then, $10(8+x)=60$

Or, $10x=-20$

Whence, $x=-2$

Two rods should therefore be *taken from* its breadth, to make the area 60 square rods.

4. Let $\frac{x}{y} =$ the fraction.

Then, $\frac{x+1}{y} = \frac{4}{9}$ (1)

and $\frac{x}{y+1} = \frac{5}{8}$ (2)

From (1), $9x - 4y = -9$ (3)

From (2), $8x - 5y = 5$ (4)

Multiplying (3) by 5, $45x - 20y = -45$ (5)

Multiplying (4) by 4, $32x - 20y = 20$ (6)

Subtracting (6) from (5), $13x = -65$

Whence, $x = -5$

Substituting in (4), $-40 - 5y = 5$

Or, $-5y = 45$

Whence, $y = -9$

Therefore, $\frac{x}{y} = \frac{-5}{-9}$, the fraction.

Hence, if the fraction be taken as $\frac{5}{9}$, 1 must be *subtracted* from the numerator and denominator, respectively, to produce the values $\frac{4}{9}$ and $\frac{5}{8}$.

5. Let $x =$ the larger number,

and $90 - x =$ the smaller number.

Then, $x - (90 - x) = 120$

Or, $2x = 210$

Whence, $x = 105$, larger no.

Therefore, $90 - x = -15$, smaller no.

If the numbers be taken as 105 and 15, their *difference* must be 90, and their *sum* 120.

6. Let $x =$ the number of years hence.

Then, $50 + x = 2(40 + x)$

Transposing, etc., $2x - x = 50 - 80$

$x = -30$, no. years.

Hence A was twice as old as B 30 *years ago.*

7. Let $x =$ B's capital,

and $3x =$ A's capital.

Then, $3x + 2000 = 2(x + 750)$

Or, $3x - 2x = 1500 - 2000$

Whence, $x = -500$, B's capital.

Therefore, $3x = -1500$, A's capital.

Hence it is evident that A's *interest* in the firm was 3 times as great as B's, but that interest was in the form of *debt*, and not of property, at the first time mentioned.

8. Let $x =$ the father's daily wages,

and $y =$ the son's daily wages.

Then, $14x + 6y = 39$ (1)

and $10x + 4y = 28$ (2)

Multiplying (2) by $\frac{3}{2}$, $15x + 6y = 42$ (3)

Equation (1), $14x + 6y = 39$

Subtracting (1) from (3), $x = 3$, father's wages.

Substituting in (2), $30 + 4y = 28$

Or, $4y = -2$

Whence, $y = -\frac{1}{2}$, son's wages.

9. Let $x =$ A's daily wages,

$y =$ B's daily wages,

and $z =$ C's daily wages.

Then, $10x + 4y + 3z = 23$ (1)

$9x + 8y + 6z = 24$ (2)

and $7x + 6y + 4z = 18$ (3)

Multiplying (1) by 2, $20x+8y+6z=46$ (4)

Subtracting (2) from (4), $11x=22$ (5)

Whence, $x=2$, A's wages.

Multiplying (1) by 4, $40x+16y+12z=92$ (6)

Multiplying (3) by 3, $\underline{21x+18y+12z=54}$ (7)

Subtracting (7) from (6), $19x-2y=38$ (8)

Substituting value of x, $38-2y=38$

Or, $2y=0$

Whence, $y=0$, B's wages.

Substituting in (1), $20+0+3z=23$

Or, $3z=3$

Whence, $z=1$, C's wages.

PROBLEM OF THE COURIERS.

(Arts. 185, 186, pp. 131–135.)

3. At 3 o'clock the hands are three hour spaces apart; hence, $a=3$. Also, $m=12$ and $n=1$.

Whence, $t=\frac{3}{12-1}$ h. $=\frac{3}{11}$ h. $=16\frac{4}{11}$ m., the time after 3 o'clock.

4. Here $a=144$, $m=12$, and $n=4$.

$$\text{Whence, } t=\frac{144}{12-4}=\frac{144}{8}=18, \text{ no. hours.}$$

A travels 18 × 12, or 216, miles.

B travels 18 × 4, or 72, miles.

5. At 12 o'clock the hands are together, and the minute-hand must therefore gain 6 hour spaces before the two hands point in opposite directions.

Hence, $a=6$, $m=12$, and $n=1$.

Therefore, $t=\frac{6}{12-1}$ h. $=\frac{6}{11}$ h. $=32\frac{8}{11}$ m.

INEQUALITIES.

(Art. 198, pp. 137–140.)

4. Given $5x - 6 > 19$

Transposing and uniting, $5x > 25$

Whence, $x > 5$

5. Given
$$\begin{cases} 2x - 5 > 25 & (1) \\ 3x - 7 < 2x + 13 & (2) \end{cases}$$

Transposing and uniting in (1), $2x > 30$

Whence, $x > 15$

Transposing and uniting in (2), $x < 20$

6. Given
$$\begin{cases} 3x + 1 > 13 - x & (1) \\ 4x - 7 < 2x + 3 & (2) \end{cases}$$

Transposing and uniting in (1), $4x > 12$

Whence, $x > 3$

Transposing and uniting in (2), $2x < 10$

Whence, $x < 5$

Therefore, $x = 4$

7. Given
$$\begin{cases} 5x + 3y > 46 - y & (1) \\ 2y - 2x = -8 & (2) \end{cases}$$

From (1), $5x + 4y > 46$ (3)

Multiplying (2) by -2, $4x - 4y = 16$ (4)

Adding (3) and (4), $9x > 62$ (5)

Whence, $x > 6\frac{8}{9}$ (6)

Multiplying (2) by $-\frac{5}{2}$, $5x - 5y = 20$ (7)

Inequality (3), $5x + 4y > 46$

Subtracting (7) from (3), $9y > 26$

Whence, $y > 2\frac{8}{9}$

8. Given
$$\begin{cases} \dfrac{cx}{3} + dx - cd > \dfrac{c^2}{3} & (1) \\[2ex] \dfrac{dx}{8} - cx + cd < \dfrac{d^2}{8} & (2) \end{cases}$$

From (1), $cx + 3dx > c^2 + 3cd$

Factoring, $(c + 3d)x > (c + 3d)c$

Whence, $x > c$

From (2), $dx - 8cx < d^2 - 8cd$

Factoring, $(d - 8c)x < (d - 8c)d$

Whence, $x < d$

9. Given $\left\{\begin{array}{l} 2x + 3y < 57 \\ 4x + 2y = 64 \end{array}\right.$ (1) (2)

Dividing (2) by 2, $2x + y = 32$ (3)

Subtracting (3) from (1), $2y < 25$ (4)

Whence, $y < 12\frac{1}{2}$ (5)

Multiplying (3) by 3, $6x + 3y = 96$ (6)

Subtracting (6) from (1), $-4x < -39$

Or, Art. 192, $4x > 39$

Whence, $x > 9\frac{3}{4}$

10. Let $x =$ the number of pupils.

Then, $2x - 7 > 29$ (1)

and $3x - 5 < 2x + 16$ (2)

From (1), $2x > 36$

Whence, $x > 18$

From (2), $x < 21$

11. Let $x =$ the number.

Then, $3x + 16 > 2x + 24$ (1)

and $\frac{2x}{5} + 5 < 11$ (2)

From (1), $x > 8$

From (2), $\frac{2x}{5} < 6$

Multiplying by $\frac{5}{2}$, $x < 15$

12. Let $x =$ the number of sheep.

Then, $3x + 2 > 2x + 61$ (1)

and $5x - 70 < 4x - 9$ (2)

From (1), $x > 59$

From (2), $x < 61$

As the number of sheep must be integral,

$$x = 60$$

INVOLUTION.

POWERS OF MONOMIALS.

(ARTS. **203–205**, pp. 141–143.)

9. See Art. 77.

11 and 23. The sign of the answer is either + or —, according as m represents an even or an odd number.

22. $(-3x^2y^{-2}z^{-1})^{-3} = -3^{-3}x^{-6}y^6z^3 = -\frac{1}{27}x^{-6}y^6z^3.$

24. $\left(\frac{-2a^{-m}x^2}{bc^{-n}y^2}\right)^4 = \frac{16a^{-4m}x^8}{b^4c^{-4n}y^8} = 16a^{-4m}b^{-4}c^{4n}x^8y^{-8}.$

25. $(-8a^2bx^{-1}y^{-3})^{-2} = 8^{-2}a^{-4}b^{-2}x^2y^6 = \frac{1}{64}a^{-4}b^{-2}x^2y^6.$

POWERS OF POLYNOMIALS.

(ART. **206**, p. 144.)

1. $(a-b)(a-b)(a-b) = a^3 - 3a^2b + 3ab^2 - b^3.$

2. $\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{b} - \frac{b}{a}\right) = \frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}.$

3. $(1+a^2+b^{-2})\,(1+a^2+b^{-2})$
$=1+2\,a^2+2\,b^{-2}+a^4+2\,a^2b^{-2}+b^{-4};$
$(1+2\,a^2+2\,b^{-2}+a^4+2\,a^2b^{-2}+b^{-4})\,(1+a^2+b^{-2})$
$=1+3\,a^2+3\,b^{-2}+3\,a^4+6\,a^2b^{-2}+3\,b^{-4}+a^6+3\,a^4b^{-2}$
$+3\,a^2b^{-4}+b^{-6}.$

4. $(a+m-n)\,(a+m-n)$
$=a^2+2\,a\,m-2\,a\,n+m^2-2\,m\,n+n^2.$

5. $(a^m-a^n)\,(a^m-a^n)=a^{2m}-2\,a^{m+n}+a^{2n};$
$(a^{2m}-2\,a^{m+n}+a^{2n})\,(a^m-a^n)$
$=a^{3m}-3\,a^{2m+n}+3\,a^{m+2n}-a^{3n};$
$(a^{3m}-3\,a^{2m+n}+3\,a^{m+2n}-a^{3n})\,(a^m-a^n)$
$=a^{4m}-4\,a^{3m+n}+6\,a^{2m+2n}-4\,a^{m+3n}+a^{4n}.$

6. $(a+b)\,(a+b)=a^2+2\,a\,b+b^2;$
$(a^2+2\,a\,b+b^2)\,(a+b)=a^3+3\,a^2b+3\,a\,b^2+b^3;$
$(a^3+3\,a^2b+3\,a\,b^2+b^3)\,(a+b);$
$=a^4+4\,a^3b+6\,a^2b^2+4\,a\,b^3+b^4;$
$(a^4+4\,a^3b+6\,a^2b^2+4\,a\,b^3+b^4)\,(a+b)$
$=a^5+5\,a^4b+10\,a^3b^2+10\,a^2b^3+5\,a\,b^4+b^5.$

POLYNOMIAL SQUARES.

(Art. 207, pp. 145, 146.)

2. $(2\,x^2+3\,x+4)^2$
$=(2x^2)^2+2\,(2x^2)\,(3x)+2\,(2x^2)\,4+(3x)^2+2\,(3x)\,4+4^2$
$=4\,x^4+12\,x^3+25\,x^2+24\,x+16.$

3 $(2\,x^2-3\,x+\frac{1}{2})^2$
$=(2x^2)^2-2\,(2x^2)\,(3x)+2\,(2x^2)\,\frac{1}{2}+(3x)^2-2\,(3x)\,\frac{1}{2}+(\frac{1}{2})^2$
$=4\,x^4-12\,x^3+11\,x^2-3\,x+\frac{1}{4}.$

5. $(x^3+2\,x^2+x+2)^2$
$=(x^3)^2+2\,x^3\,(2\,x^2)+2\,x^3x+2\,x^3\,2+(2\,x^2)^2+2\,(2\,x^2)\,x$
$+2\,(2\,x^2)\,2+x^2+2\,x\,2+2^2$

$$=x^6+4x^5+2x^4+4x^3+4x^4+4x^3+8x^2+x^2+4x+4$$
$$=x^6+4x^5+6x^4+8x^3+9x^2+4x+4.$$

6. $(1-2x+3x^2)^2$
$$=1-2(2x)+2(3x^2)+(2x)^2-2(2x)(3x^2)+(3x^2)^2$$
$$=1-4x+10x^2-12x^3+9x^4.$$

7. $(1+x+x^2+x^3)^2$
$$=1+2x+2x^2+2x^3+x^2+2xx^2+2xx^3+(x^2)^2$$
$$+2x^2x^3+(x^3)^2$$
$$=1+2x+3x^2+4x^3+3x^4+2x^5+x^6.$$

8. The answer is obtained by a direct application of the rule, as in the 1st and 4th.

POLYNOMIAL CUBES.

(Art. 208, pp. 146, 147.)

2. $(x^2-x-1)^3=(x^2)^3-3(x^2)^2x-3(x^2)^2-x^3+3x^2x^2$
$$-3x^2-1+3x^2-3x+6x^2x$$
$$=x^6-3x^5+5x^3-3x-1.$$

3. $(a-b+1)^3=a^3-3a^2b+3a^2-b^3+3ab^2+3b^2+$
$$1+3a-3b-6ab$$
$$=a^3-3a^2b+3a^2+3ab^2-6ab+3a-b^3+3b^2-$$
$$3b+1.$$

4. $(1+x+x^2+x^3)^3=1+3x+3x^2+3x^3+x^3+3x^2+$
$$3x^2x^2+3x^2x^3+(x^2)^3+3(x^2)^2+3(x^2)^2x+3(x^2)^2x^3$$
$$+(x^3)^3+3(x^3)^2+3(x^3)^2x+3(x^3)^2x^2+6xx^2+6xx^3$$
$$+6x^2x^3+6xx^2x^3$$
$$=1+3x+3x^2+3x^2+3x^3+x^3+6x^3+3x^4+3x^4$$
$$+6x^4+3x^5+3x^5+6x^5+x^6+3x^6+6x^6+3x^7+$$
$$3x^7+3x^8+x^9$$

$$= 1 + 3x + 6x^2 + 10x^3 + 12x^4 + 12x^5 + 10x^6 + 6x^7 + 3x^8 + x^9.$$

5. The answer is obtained by a direct application of the rule, writing the cubes first, and remembering that the squares of the third and fourth terms are $4c^4$ and d^6.

EVOLUTION.

SQUARE ROOT OF NUMBERS.

(Arts. **216–219**, pp. 150–154.)

(2.)

$$\begin{array}{r|l|l}
 & 2\dot{7}3\dot{5}2\dot{9} & 523 \\
 & 25 & \\ \hline
102 & 235 & \\
 & 204 & \\ \hline
1043 & 3129 & \\
 & 3129 & \\ \hline
\end{array}$$

(3.)

$$\begin{array}{r|l|l}
 & \dot{4}5\dot{7}9\dot{6} & 214 \\
 & 4 & \\ \hline
41 & 57 & \\
 & 41 & \\ \hline
424 & 1696 & \\
 & 1696 & \\ \hline
\end{array}$$

(4.)

$$\begin{array}{r|l|l}
 & 1\dot{0}6\dot{9}2\dot{9} & 327 \\
 & 9 & \\ \hline
62 & 169 & \\
 & 124 & \\ \hline
647 & 4529 & \\
 & 4529 & \\ \hline
\end{array}$$

(5.)

$$\begin{array}{r|l|l}
 & 3\dot{3}.1\dot{7}7\dot{6} & 5.76 \\
 & 25 & \\ \hline
107 & 817 & \\
 & 749 & \\ \hline
1146 & 6876 & \\
 & 6876 & \\ \hline
\end{array}$$

(6.)

$$\begin{array}{r|l|l}
 & .9\dot{4}0\dot{9} & .97 \\
 & 81 & \\ \hline
187 & 1309 & \\
 & 1309 & \\ \hline
\end{array}$$

(7.)

$$\begin{array}{r|l|l}
 & 6\dot{5}6\dot{1} & 81 \\
 & 64 & \\ \hline
161 & 161 & \\
 & 161 & \\ \hline
\end{array}$$

(Continued on next page.)

$$\begin{array}{r|l|l}
 & 9\dot{0}2\dot{5} & 95 \\
 & 81 & \\ \hline
185 & 925 & \\
 & 925 & \\ \hline
\end{array}$$

(8.)

$$\sqrt{5\tfrac{11}{49}} = \sqrt{\tfrac{256}{49}} = \tfrac{16}{7} = 2\tfrac{2}{7}$$

(9.)

$$\begin{array}{r|l|l}
 & \dot{2}.0\dot{0}0\dot{0}0\dot{0} & 1.414+ \\
 & 1 & \\ \hline
24 & 100 & \\
 & 96 & \\ \hline
281 & 400 & \\
 & 281 & \\ \hline
2824 & 11900 & \\
 & 11296 & \\ \hline
 & 604 &
\end{array}$$

(11.)

$$7 \times 10^2 = 700$$
$$\sqrt{700} = 26+$$
$$26 \div 10 = 2\tfrac{6}{10}$$

(12.)

$$11 \times 15^2 = 2475$$
$$\sqrt{2475} = 50-$$
$$50 \div 15 = 3\tfrac{5}{15}$$

(13.)

$$\begin{array}{r|l|l}
 & 1\dot{2}.0\dot{0}0\dot{0} & 3.46 \\
 & 9 & \\ \hline
64 & 300 & \\
 & 256 & \\ \hline
686 & 4400 & \\
 & 4116 & \\ \hline
 & 284 &
\end{array}$$

(14.)

$$\begin{array}{r|l|l}
 & \dot{5}.0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0} & 2.23607 \\
 & 4 & \\ \hline
42 & 100 & \\
 & 84 & \\ \hline
443 & 1600 & \\
 & 1329 & \\ \hline
4466 & 27100 & \\
 & 26796 & \\ \hline
447207 & 3040000 & \\
 & 3130449 & \\ \hline
\end{array}$$

The last two examples may also be solved like the 11th and 12th.

17. $\frac{\sqrt{2}}{\sqrt{49}} = \frac{1.41421}{7} = .20203.$

20. $\frac{5}{6} \times \frac{384}{384} = \frac{1920}{2304}; \; \frac{\sqrt{1920}}{\sqrt{2304}} = \frac{44}{48}$, nearly.

(21.)

$$\tfrac{11}{17} = .64705882$$

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & .6\dot{4}7\dot{0}5\dot{8}8\dot{2} \;|\; .8044 \\
\multicolumn{1}{r}{} & 64 \\
\hline
1604 & 7058 \\
 & 6416 \\
\hline
16084 & 64282 \\
 & 64336 \\
\hline
\end{array}$$

CUBE ROOT OF NUMBERS.

(Art. 222, pp. 155–158.)

(3.)

$$\begin{array}{r|l}
 & \dot{1}86\dot{0}86\dot{7} \;|\; 123 \\
 & 1 \\
\hline
300 & 860 \\
60 & \\
4 & \\
\overline{364} \times 2 = & 728 \\
\cline{2-2}
43200 & 132867 \\
1080 & \\
9 & \\
\overline{44289} \times 3 = & 132867 \\
\cline{2-2}
\end{array}$$

(4.)

$$\begin{array}{rl}
 & .72\dot{4}15\dot{0}79\dot{2}\;|\;.898\\
 & \underline{512}\\
19200 & 212150\\
2160 & \\
\underline{81} & \\
21441 \times 9 = & \underline{192969}\\
2376300 & 19181792\\
21360 & \\
\underline{64} & \\
2397724 \times 8 = & \underline{19181792}
\end{array}$$

(5.)

$$\begin{array}{rl}
 & \dot{1}48\dot{1}.54\dot{4}\;|\;11.4\\
 & \underline{1}\\
300 & 481\\
30 & \\
\underline{1} & \\
331 \times 1 = & \underline{331}\\
36300 & 150544\\
1320 & \\
\underline{16} & \\
37636 \times 4 = & \underline{150544}
\end{array}$$

(6.)

$$\begin{array}{rl}
 & 2\dot{9}79\dot{1}\;|\;31\\
 & \underline{27}\\
2700 & 2791\\
90 & \\
\underline{1} & \\
2791 \times 1 = & \underline{2791}
\end{array}$$

$$\begin{array}{rl}
 & 68\dot{1}47\dot{2}\;|\;88\\
 & \underline{512}\\
19200 & 169472\\
1920 & \\
\underline{64} & \\
21184 \times 8 = & \underline{169472}
\end{array}$$

(7.)

$$\tfrac{3}{17} = .176470588235$$

$$\begin{array}{rr|l}
 & .17\dot{6}47\dot{0}58\dot{8}23\dot{5} & .5609+ \\
 & 125 & \\
7500 & 51470 & \\
900 & & \\
36 & & \\
\hline
8436 \times 6 = & 50616 & \\
\hline
94080000 & 854588235 & \\
151200 & & \\
81 & & \\
\hline
94231281 \times 9 = & 848081529 & \\
\hline
 & 6506706 &
\end{array}$$

ROOTS OF MONOMIALS.

(ART. **223**, pp. 158–160.)

7. Divide each exponent by m.

8. Extract the cube root of -8, and divide each exponent by 3.

9. Extract the square root of 16, and multiply each exponent by $\frac{1}{2}$, or divide by 2.

10. Multiply each exponent by $\frac{1}{n}$, or divide by n.

11. $\frac{1}{a^n}(x-y)^{\frac{m}{n}} = a^{-n}(x-y)^{\frac{m}{n}}$; dividing each exponent by m we obtain $a^{-\frac{n}{m}}(x-y)^{\frac{1}{n}}$.

12. Divide each exponent by 2, and prefix the double sign, because it is an *even* root.

SQUARE ROOT OF POLYNOMIALS.

(Arts. **224–226**, pp. 160–163.)

(3.)

$$\begin{array}{r|l|l}
 & 4x^4-4x^3-3x^2+2x+1 & 2x^2-x-1 \\
 & 4x^4 & \\ \hline
4x^2-x & -4x^3-3x^2 & \\
 & -4x^3+x^2 & \\ \hline
4x^2-2x-1 & -4x^2+2x+1 & \\
 & -4x^2+2x+1 & \\ \hline
\end{array}$$

(4.)

$$\begin{array}{r|l|l}
 & 4a^4-16a^3+24a^2-16a+4 & 2a^2-4a+2 \\
 & 4a^4 & \\ \hline
4a^2-4a & -16a^3+24a^2 & \\
 & -16a^3+16a^2 & \\ \hline
4a^2-8a+2 & 8a^2-16a+4 & \\
 & 8a^2-16a+4 & \\ \hline
\end{array}$$

(5.)

$$\begin{array}{r|l|l}
 & m^2+2m-1-\frac{2}{m}+\frac{1}{m^2} & m+1-\frac{1}{m} \\
 & m^2 & \\ \hline
2m+1 & 2m-1 & \\
 & 2m+1 & \\ \hline
2m+2-\frac{1}{m} & -2-\frac{2}{m}+\frac{1}{m^2} & \\
 & -2-\frac{2}{m}+\frac{1}{m^2} & \\ \hline
\end{array}$$

(6.)

$$(x+x^{-1})^2-4\,(x-x^{-1})=x^2+2+x^{-2}-4x+4x^{-1}$$

$$\begin{array}{r|l|l}
 & x^2-4x+2+4x^{-1}+x^{-2} & x-2-x^{-1} \\
 & x^2 & \\
\hline
2x-2 & -4x+2 & \\
 & -4x+4 & \\
\hline
2x-4-x^{-1} & -2+4x^{-1}+x^{-2} & \\
 & -2+4x^{-1}+x^{-2} & \\
\hline
\end{array}$$

(7.)

$$\begin{array}{r|l|l}
 & 1+x & 1+\frac{x}{2}-\frac{x^2}{8}+\text{etc.} \\
 & 1 & \\
\hline
2+\frac{x}{2} & x & \\
 & x+\frac{x^2}{4} & \\
\hline
2+x-\frac{x^2}{8} & -\frac{x^2}{4} & \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} & \\
\hline
 & \frac{x^3}{8}-\frac{x^4}{64} & \\
\end{array}$$

(8.)

$$\begin{array}{r|l|l}
 & a+b & a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\text{etc.} \\
 & a & \\
\hline
2a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}} & b & \\
 & b+\frac{b^2}{4a} & \\
\hline
2a^{\frac{1}{2}}+\frac{b}{a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}} & -\frac{b^2}{4a} & \\
 & -\frac{b^2}{4a}-\frac{b^3}{8a^2}+\frac{b^4}{64a^3} & \\
\hline
 & \frac{b^3}{8a^2}-\frac{b^4}{64a^3} & \\
\end{array}$$

(9.)

$$\begin{array}{r|l}
1-x & 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\text{etc.} \\
1 & \\
\hline
2-\frac{x}{2} & -x \\
 & -x+\frac{x^2}{4} \\
\hline
2-x-\frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2-x-\frac{x^2}{4}-\frac{x^3}{16} & -\frac{x^3}{8}-\frac{x^4}{64} \\
 & -\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{256}
\end{array}$$

(10.)

$$\begin{array}{r|l}
a^2+x^2 & a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\text{etc.} \\
a^2 & \\
\hline
2a+\frac{x^2}{2a} & x^2 \\
 & x^2+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3} & -\frac{x^4}{4a^2} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{16a^5} & \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{8a^5}-\frac{5x^8}{128a^7} & -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}} \\
 & -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8}+\frac{5x^{12}}{512a^{10}}-\text{etc.} \\
\hline
 & \frac{7x^{10}}{128a^8}-\frac{7x^{12}}{512a^{10}}+\text{etc.}
\end{array}$$

11. $(9y^4 - 6x^3y^2 + x^6)^{\frac{1}{2}} = \sqrt{9y^4} - \sqrt{x^6} = 3y^2 - x^3$

12. $\left(a^2 - ax + \frac{x^2}{4}\right)^{\frac{1}{2}} = \sqrt{a^2} - \sqrt{\frac{x^2}{4}} = a - \frac{x}{2}$

13. $(a^{2m} - 4a^{m+n} + 4a^{2n})^{\frac{1}{2}} = \sqrt{a^{2m}} - \sqrt{4a^{2n}} = a^m - 2a^n$

14. $\left(\frac{a^2}{b^2} - \frac{4ab}{3bc} + \frac{4b^2}{9c^2}\right)^{\frac{1}{2}} = \sqrt{\frac{a^2}{b^2}} - \sqrt{\frac{4b^2}{9c^2}} = \frac{a}{b} - \frac{2b}{3c}$

CUBE ROOT OF POLYNOMIALS.

(ART. 227, pp. 163–166.)

(3.)

$$\begin{array}{r|l|l}
 & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 & a+b+c \\
 & a^3 & \\ \hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
 & 3a^2b+3ab^2+b^3 & \\ \hline
3a^2+6ab+3b^2+3ac+3bc+c^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 & \\
 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 & \\ \hline
\end{array}$$

(4.)

$$\begin{array}{r|l|l}
 & 1-6y+12y^2-8y^3 & 1-2y \\
 & 1 & \\ \hline
3-6y+4y^2 & -6y+12y^2-8y^3 & \\
 & -6y+12y^2-8y^3 & \\ \hline
\end{array}$$

(5.)

$$\begin{array}{r|l|l}
 & x^3+3x+\frac{3}{x}+\frac{1}{x^3} & x+\frac{1}{x} \\
 & x^3 & \\ \hline
3x^2+3+\frac{1}{x^2} & 3x+\frac{3}{x}+\frac{1}{x^3} & \\
 & 3x+\frac{3}{x}+\frac{1}{x^3} & \\ \hline
\end{array}$$

(6.)

$$\begin{array}{r|l|l}
 & \frac{a^6}{b^2}-6a^4+12a^2b^2-8b^4 & \frac{a^2}{b^{\frac{2}{3}}}-2b^{\frac{4}{3}} \\
 & \frac{a^6}{b^2} & \\
\hline
\frac{3a^4}{b^{\frac{4}{3}}}-6a^2b^{\frac{2}{3}}+4b^{\frac{8}{3}} & -6a^4+12a^2b^2-8b^4 & \\
 & -6a^4+12a^2b^2-8b^4 & \\
\hline
\end{array}$$

(7.)

$$\begin{array}{r|l|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & x^2-2x+1 \\
 & x^6 & \\
\hline
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 & \\
 & -6x^5+12x^4-\ 8x^3 & \\
\hline
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1 & \\
 & 3x^4-12x^3+15x^2-6x+1 & \\
\hline
\end{array}$$

ANY ROOT OF POLYNOMIALS.

(ARTS. **228, 229**, pp. 166–168.)

(2.)

$$\begin{array}{r|l|l}
 & m^6-6m^5+40m^3-96m-64 & m^2-2m-4 \\
(m^2)^3= & m^6 & \\
\hline
3m^4 & -6m^5 & \\
\hline
(m^2-2m)^3= & m^6-6m^5+12m^4-8m^3 & \\
\hline
3m^4 & -12m^4 & \\
\hline
(m^2-2m-4)^3= & m^6-6m^5+40m^3-96m-64 & \\
\end{array}$$

(3.)

$$\begin{array}{r|l}
a^4-2a^3x+3a^2x^2-2ax^3+x^4 & a^2-ax+x^2 \\
a^4 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
2a^2 & -2a^3x \\ \hline
\end{array}$$

$$(a^2-ax)^2=a^4-2a^3x+a^2x^2$$

$$\begin{array}{r|l}
2a^2 & 2a^2x^4 \\ \hline
\end{array}$$

$$(a^2-ax+x^2)^2=a^4-2a^3x+3a^2x^2-2ax^3+x^4$$

(4.)

$$\begin{array}{r|l}
32x^5-80x^4+80x^3-40x^2+10x-1 & 2x-1 \\
32x^5 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
80x^4 & -80x^4 \\ \hline
\end{array}$$

$$(2x-1)^5=32x^5-80x^4+80x^3-40x^2+10x-1$$

(5.)

$$\begin{array}{r|l}
16x^4-96x^3y+216x^2y^2-216xy^3+81y^4 & 4x^2-12xy+9y^2 \\
16x^4 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
8x^2-12xy & -96x^3y+216x^2y^2 \\
 & -96x^3y+144x^2y^2 \\ \hline
8x^2-24xy+9y^2 & 72x^2y^2-216xy^3+81y^4 \\
 & 72x^2y^2-216xy^3+81y^4 \\ \hline
\end{array}$$

$$\begin{array}{r|l}
4x^2-12xy+9y^2 & 2x-3y \\
4x^2 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
4x-3y & -12xy+9y^2 \\
 & -12xy+9y^2 \\ \hline
\end{array}$$

(6.)

$$\begin{array}{l|l}
a^6-6a^4+15a^2-20+\frac{15}{a^2}-\frac{6}{a^4}+\frac{1}{a^6} & a^3-3a+\frac{3}{a}-\frac{1}{a^3} \\
a^6 &
\end{array}$$

$$\begin{array}{r|l}
2a^3-3a & -6a^4+15a^2 \\
 & -6a^4+9a^2 \\ \hline
2a^3-6a+\frac{3}{a} & 6a^2-20+\frac{15}{a^2} \\
 & 6a^2-18+\frac{9}{a^2} \\ \hline
2a^3-6a+\frac{6}{a}-\frac{1}{a^3} & -2+\frac{6}{a^2}-\frac{6}{a^4}+\frac{1}{a^6} \\
 & -2+\frac{6}{a^2}-\frac{6}{a^4}+\frac{1}{a^6} \\ \hline
\end{array}$$

$$\begin{array}{l|l}
a^3-3a+\frac{3}{a}-\frac{1}{a^3} & a-\frac{1}{a} \\
a^3 &
\end{array}$$

$$\begin{array}{r|l}
3a^2-3+\frac{1}{a^2} & -3a+\frac{3}{a}-\frac{1}{a^3} \\
 & -3a+\frac{3}{a}-\frac{1}{a^3} \\ \hline
\end{array}$$

RADICALS.

REDUCTION OF RADICALS.

CASE I.

(ARTS. **236**, **237**, pp. 169–172.)

2. $\sqrt{27a^3x^5}=\sqrt{9a^2x^4\times 3ax}=3ax^2\sqrt{3ax}.$

3. $(189a^4b^3c^2)^{\frac{1}{3}}=(27a^3b^3\times 7ac^2)^{\frac{1}{3}}=3ab(7ac^2)^{\frac{1}{3}}.$

4. $\sqrt{96a^2x^3}=\sqrt{16a^2x^2\times 6x}=4ax\sqrt{6x}.$

5. $\sqrt[n]{a^{m+n}b^n} = \sqrt[n]{a^n b^n \times a^m} = a\,b\,\sqrt[n]{a^m}.$

6. $2\,c\,\sqrt[4]{768} = 2\,c\,\sqrt[4]{256 \times 3} = 2\,c \times 4\,\sqrt[4]{3} = 8\,c\,\sqrt[4]{3}.$

7. $\sqrt{a^3 - a^2 b} = \sqrt{a^2(a-b)} = a\,\sqrt{a-b}.$

8. $(a\,x^2 - 6\,ax + 9\,a)^{\frac{1}{2}} = [(x-3)^2 \times a]^{\frac{1}{2}} = (x-3)\,a^{\frac{1}{2}}.$

9. $\sqrt{(x^2-y^2)\,(x+y)} = \sqrt{(x+y)^2\,(x-y)} = (x+y)\,\sqrt{x-y}.$

12. $\frac{1}{2}\sqrt{\frac{x^3}{15}} = \frac{1}{2}\sqrt{\frac{x^3}{15} \times \frac{15}{15}} = \frac{1}{2}\sqrt{\frac{15\,x^3}{225}} = \frac{1}{2}\sqrt{\frac{x^2}{225} \times 15\,x}$

$= \frac{1}{2} \times \frac{x}{15}\sqrt{15\,x} = \frac{x}{30}\sqrt{15\,x}.$

13. $\frac{3}{11}\sqrt{\frac{4}{7}} = \frac{3}{11}\sqrt{\frac{4}{7} \times \frac{7}{7}} = \frac{3}{11}\sqrt{\frac{4}{49} \times 7} = \frac{3}{11} \times \frac{2}{7}\sqrt{7} = \frac{6}{77}\sqrt{7}.$

14. $\sqrt{\frac{ab^2}{4\,(a+x)}} = \sqrt{\frac{a\,b^2}{4\,(a+x)} \times \frac{a+x}{a+x}}$

$= \sqrt{\frac{b^2}{4\,(a+x)^2} \times a\,(a+x)} = \frac{b}{2\,(a+x)}\sqrt{a\,(a+x)}.$

15. $\frac{a}{a^2-b^2}\sqrt{\frac{a^3 b - 2\,a^2 b^2 + a\,b^3}{b^2}} = \frac{a}{a^2-b^2}\sqrt{\frac{(a-b)^2}{b^2} \times a\,b}$

$= \frac{a\,(a-b)}{b\,(a^2-b^2)}\sqrt{a\,b} = \frac{a}{b\,(a+b)}\sqrt{a\,b}.$

CASE II.

(ARTS. **238**, **239**, pp. 172, 173.)

4. $\frac{x^2}{3} = \sqrt[3]{\left(\frac{x^2}{3}\right)^3} = \sqrt[3]{\frac{x^6}{27}}.$

5. $\frac{x^2}{x-y} = \sqrt[4]{\left(\frac{x^2}{x-y}\right)^4} = \sqrt[4]{\frac{x^8}{(x-y)^4}}.$

7. $x\,(2a-x)^{\frac{1}{2}} = [x^2\,(2\,a-x)]^{\frac{1}{2}} = (2\,a\,x^2 - x^3)^{\frac{1}{2}}.$

9. $\frac{1}{2}\sqrt[4]{b^2} = \sqrt[4]{(\frac{1}{2})^4 b^2} = \sqrt[4]{\frac{b^2}{16}}$, or $\sqrt{\frac{b}{4}}$.

10. $\frac{2x}{3}\sqrt[3]{\frac{9}{4x^2}} = \sqrt[3]{\frac{8x^3}{27} \times \frac{9}{4x^2}} = \sqrt[3]{\frac{2x}{3}}$.

11. $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{(x+1)^2(x-1)}{(x-1)^2(x+1)}} = \sqrt{\frac{x+1}{x-1}}$.

CASE III.

(Art. 240, pp. 174, 175.)

2. $a^2 = a^{\frac{8}{4}} = \sqrt[4]{a^8}$; $a^{\frac{1}{2}} = a^{\frac{2}{4}} = \sqrt[4]{a^2}$.

3. $a^{\frac{1}{2}} = a^{\frac{2}{4}} = \sqrt[4]{a^2}$; $b^{\frac{1}{4}} = \sqrt[4]{b}$.

4. $x^{\frac{1}{m}} = x^{\frac{n}{mn}} = \sqrt[mn]{x^n}$; $y^{\frac{1}{n}} = y^{\frac{m}{mn}} = \sqrt[mn]{y^m}$.

5. $\sqrt[4]{a} = \sqrt[24]{a^6}$; $\sqrt[6]{5b} = \sqrt[24]{(5b)^4} = \sqrt[24]{625b^4}$;

$\sqrt[8]{3c} = \sqrt[24]{(3c)^3} = \sqrt[24]{27c^3}$.

6. In the first, multiply the index and the exponent by 2; in the second, by 3.

ADDITION OF RADICALS.

(Art. 241, pp. 175, 176.)

3. $\sqrt[3]{375} = \sqrt[3]{125 \times 3} = 5\sqrt[3]{3}$

$\sqrt[3]{192} = \sqrt[3]{64 \times 3} = 4\sqrt[3]{3}$

Sum $= 9\sqrt[3]{3}$

4. $$3\sqrt{a^2b} = 3a\sqrt{b}$$
$$5\sqrt{16a^4b} = 5 \times 4a^2\sqrt{b} = 20a^2\sqrt{b}$$
$$\text{Sum} = (3a + 20a^2)\sqrt{b}$$

5. $$3\sqrt{\tfrac{2}{5}} = 3\sqrt{\tfrac{2}{5} \times \tfrac{5}{5}} = 3\sqrt{\tfrac{1}{25} \times 10} = \tfrac{3}{5}\sqrt{10}$$
$$2\sqrt{\tfrac{1}{10}} = 2\sqrt{\tfrac{1}{10} \times \tfrac{10}{10}} = 2\sqrt{\tfrac{1}{100} \times 10} = \tfrac{1}{5}\sqrt{10}$$
$$\text{Sum} = \tfrac{4}{5}\sqrt{10}$$

6. $$(b^4y)^{\frac{1}{3}} = (b^3 \times by)^{\frac{1}{3}} = b(by)^{\frac{1}{3}}$$
$$(by^4)^{\frac{1}{3}} = (y^3 \times by)^{\frac{1}{3}} = y(by)^{\frac{1}{3}}$$
$$\text{Sum} = (b + y)(by)^{\frac{1}{3}}$$

7. $$8\sqrt{\tfrac{3}{4}} = 8\sqrt{\tfrac{1}{4} \times 3} = \tfrac{8}{2}\sqrt{3} = 4\sqrt{3}$$
$$-\tfrac{1}{2}\sqrt{12} = -\tfrac{1}{2}\sqrt{4 \times 3} = -\tfrac{2}{2}\sqrt{3} = -\sqrt{3}$$
$$4\sqrt{27} = 4\sqrt{9 \times 3} = 4 \times 3\sqrt{3} = 12\sqrt{3}$$
$$-2\sqrt{\tfrac{3}{16}} = -2\sqrt{\tfrac{1}{16} \times 3} = -\tfrac{2}{4}\sqrt{3} = -\tfrac{1}{2}\sqrt{3}$$
$$\text{Sum} = 14\tfrac{1}{2}\sqrt{3}$$

8. $$\sqrt{48ab^2} = \sqrt{16b^2 \times 3a} = 4b\sqrt{3a}$$
$$\sqrt{75a^3} = \sqrt{25a^2 \times 3a} = 5a\sqrt{3a}$$
$$\sqrt{3a(a-9b)^2} = (a-9b)\sqrt{3a}$$
$$\text{Sum} = (6a - 5b)\sqrt{3a}$$

9. $$\sqrt[3]{\frac{27a^5d}{2b}} = \sqrt[3]{27a^3 \times \frac{a^2d}{2b}} = 3a\sqrt[3]{\frac{a^2d}{2b}}$$
$$-\sqrt[3]{\frac{a^2d}{2b}}$$
$$\text{Sum} = (3a - 1)\sqrt[3]{\frac{a^2d}{2b}}$$

10. $3(242a^5b^5)^{\frac{1}{2}} = 3(121a^4b^4 \times 2ab)^{\frac{1}{2}} = 3 \times 11a^2b^2(2ab)^{\frac{1}{2}} = 33a^2b^2(2ab)^{\frac{1}{2}}$

$\underline{11(2a^3b^3)^{\frac{1}{2}} = 11(a^2b^2 \times 2ab)^{\frac{1}{2}} = 11ab(2ab)^{\frac{1}{2}}}$

Sum $= 11(3a^2b^2 + ab)(2ab)^{\frac{1}{2}} = (33a^2b^2 + 11ab)(2ab)^{\frac{1}{2}}$

11. $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$

$2\sqrt{72} = 2\sqrt{36 \times 2} = 2 \times 6\sqrt{2} = 12\sqrt{2}$

$\underline{a\sqrt{bx^2} = ax\sqrt{b}}$

Sum $= 2\sqrt{6} + 12\sqrt{2} + ax\sqrt{b}$

12. $\sqrt[3]{125b^4 - 250ab^3} = \sqrt[3]{125b^3(b-2a)} = 5b\sqrt[3]{b-2a}$

$\sqrt[3]{64a^3b - 128a^4} = \sqrt[3]{64a^3(b-2a)} = \underline{4a\sqrt[3]{b-2a}}$

Sum $= (5b + 4a)\sqrt[3]{b-2a}$

SUBTRACTION OF RADICALS.

(ART. **242**, pp. 176, 177.)

2. $2\sqrt[3]{320} = 2\sqrt[3]{64 \times 5} = 2 \times 4\sqrt[3]{5} = 8\sqrt[3]{5}$

$3\sqrt[3]{40} = 3\sqrt[3]{8 \times 5} = 3 \times 2\sqrt[3]{5} = \underline{6\sqrt[3]{5}}$

Difference $= 2\sqrt[3]{5}$

3. $\sqrt{4ax^2} = 2x\sqrt{a}$

$3x\sqrt{9a} = 3x \times 3\sqrt{a} = \underline{9x\sqrt{a}}$

Difference $= -7x\sqrt{a}$

4. $\sqrt[3]{\tfrac{3}{5}} = \sqrt[3]{\tfrac{3}{5} \times \tfrac{25}{25}} = \sqrt[3]{\tfrac{1}{125} \times 75} = \tfrac{1}{5}\sqrt[3]{75} = \tfrac{3}{15}\sqrt[3]{75}$

$\sqrt[3]{\tfrac{25}{9}} = \sqrt[3]{\tfrac{25}{9} \times \tfrac{3}{3}} = \sqrt[3]{\tfrac{1}{27} \times 75} = \tfrac{1}{3}\sqrt[3]{75} = \underline{\tfrac{5}{15}\sqrt[3]{75}}$

Difference $= \tfrac{2}{15}\sqrt[3]{75}$

5. $$\tfrac{3}{7}\sqrt[3]{a^3 b} = \frac{3a}{7}\sqrt[3]{b}$$

$$\tfrac{2}{5}\sqrt[3]{a^6 b} = \frac{2a^2}{5}\sqrt[3]{b}$$

$$\text{Difference} = \left(\frac{3a}{7} - \frac{2a^2}{5}\right)\sqrt[3]{b}$$

6. The radicals are dissimilar, and no simplification is possible, except to separate the factor 4.

7. $$3b(2a^5b^2)^{\frac{1}{3}} = 3b(a^3 \times 2a^2b^2)^{\frac{1}{3}} = 3ab(2a^2b^2)^{\frac{1}{3}}$$

$$-7(2a^5b^5)^{\frac{1}{3}} = -7(a^3b^3 \times 2a^2b^2)^{\frac{1}{3}} = -7ab(2a^2b^2)^{\frac{1}{3}}$$

$$8a(2a^2b^5)^{\frac{1}{3}} = 8a(b^3 \times 2a^2b^2)^{\frac{1}{3}} = 8ab(2a^2b^2)^{\frac{1}{3}}$$

$$\text{Difference} = 4ab(2a^2b^2)^{\frac{1}{3}}$$

8. $$m\sqrt{1+\left(\frac{n}{m}\right)^{\frac{1}{2}}} = m\sqrt{\frac{m^{\frac{1}{2}}+n^{\frac{1}{2}}}{m^{\frac{1}{2}}}} = \frac{m}{m^{\frac{1}{4}}}\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}} = m^{\frac{3}{4}}\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}}$$

$$n\sqrt{1+\left(\frac{m}{n}\right)^{\frac{1}{2}}} = n\sqrt{\frac{n^{\frac{1}{2}}+m^{\frac{1}{2}}}{n^{\frac{1}{2}}}} = \frac{n}{n^{\frac{1}{4}}}\sqrt{n^{\frac{1}{2}}+m^{\frac{1}{2}}} = n^{\frac{3}{4}}\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}}$$

$$\text{Difference} = \left(m^{\frac{3}{4}} - n^{\frac{3}{4}}\right)\sqrt{m^{\frac{1}{2}}+n^{\frac{1}{2}}}$$

MULTIPLICATION OF RADICALS.

(ARTS. **243**, **244**, pp. 178–180.)

2. $\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n} \times \sqrt[mn]{b^m} = \sqrt[mn]{a^n b^m}$.

3. $5a^{\frac{1}{2}} \times 3a^{\frac{1}{3}} = 5\sqrt[6]{a^3} \times 3\sqrt[6]{a^2} = 15\sqrt[6]{a^5}$.

4. $4\sqrt{12} \times 3\sqrt{2} = 12\sqrt{24} = 12\sqrt{4 \times 6} = 12 \times 2\sqrt{6}$
$= 24\sqrt{6}$.

$$\frac{3}{8}\sqrt[3]{\frac{5}{9}} = \frac{3}{8}\sqrt[3]{\frac{15}{27}} = \frac{3}{8}\times\frac{1}{3}\sqrt[3]{15} = \frac{1}{8}\sqrt[3]{15}.$$

$$-b)^{\frac{3}{4}} = \sqrt[12]{(a+b)^4}\times\sqrt[12]{(a+b)^9}$$

$$c = \frac{1}{b^2}\sqrt[12]{\frac{1}{a^9}}\times c\sqrt[12]{a^{10}b^6} = \frac{c}{b^2}\sqrt[12]{ab^6}.$$

$$\frac{8}{2b} = 2a^2\sqrt{\frac{9a^2}{16b^2}} = 2a^2\times\frac{3a}{4b} = \frac{3a^3}{2b}.$$

(10.)

$$\begin{array}{r}
-\sqrt{xy}+y \\
-\sqrt{y} \\
\hline
-x\sqrt{y}+y\sqrt{x} \\
x\sqrt{y}-y\sqrt{x}+y\sqrt{y} \\
\hline
+y\sqrt{y}
\end{array}$$

$$a-c\sqrt[3]{b}) = a-c^2\sqrt[3]{b^2}.$$

(12.)

$$\begin{array}{lll}
b+\sqrt{c} & & \\
b-\sqrt{c} & & \\
\hline
b+\sqrt{ac} & & \\
b & +b+\sqrt{bc} & \\
\quad-\sqrt{ac} & -\sqrt{bc} & -c \\
\hline
b & +b & -c
\end{array}$$

DIVISION OF RADICALS.

(ART. **245**, pp. 180, 181.)

2. $\frac{6\sqrt{96}}{3\sqrt{8}} = 2\sqrt{12} = 2\sqrt{4 \times 3} = 2 \times 2\sqrt{3} = 4\sqrt{3}.$

3. $\frac{4\sqrt[3]{72}}{2\sqrt[3]{18}} = 2\sqrt[3]{4}.$

4. $\frac{\sqrt{96\,a^2}}{4\,a} = \frac{\sqrt{16\,a^2 \times 6}}{4\,a} = \frac{4\,a}{4\,a}\sqrt{6} = \sqrt{6}.$

5. Subtracting the exponent $\frac{1}{2}$ from the exponent $\frac{1}{4}$, we have the exponent $-\frac{1}{4}$. (Art. 77.)

Or, $\frac{(x^2 y^3)^{\frac{1}{4}}}{(x^2 y^3)^{\frac{1}{2}}} = \frac{(x^2 y^3)^{\frac{1}{4}}}{(x^4 y^6)^{\frac{1}{4}}} = \left(\frac{x^2 y^3}{x^4 y^6}\right)^{\frac{1}{4}} = \left(\frac{1}{x^2 y^3}\right)^{\frac{1}{4}} = (x^2 y^3)^{\;\frac{1}{4}}$

6. $\frac{2\sqrt[3]{b\,c}}{3\sqrt{a\,c}} = \frac{2\sqrt[6]{b^2 c^2}}{3\sqrt[6]{a^3 c^3}} = \frac{2}{3}\sqrt[6]{\frac{b^2}{a^3 c}} = \frac{2}{3}\sqrt[6]{\frac{a^3 b^2 c^5}{a^6 c^6}} = \frac{2}{3\,a\,c}\sqrt[6]{a^3 b^2 c^5}$

7. $\frac{b\,c\,(a^3 x^2)^{\frac{1}{n}}}{c\,(a\,x)^{\frac{1}{n}}} = b\,(a^2 x)^{\frac{1}{n}}$

8. $\frac{\sqrt{a\,x^3}}{\sqrt{b\,x}} = \sqrt{\frac{a\,x^2}{b}} = \sqrt{\frac{a\,b\,x^2}{b^2}} = \frac{x}{b}\sqrt{a\,b}$

9. $\sqrt[3]{\frac{a\,x}{2\,b\,y^2}} \div \sqrt[6]{\frac{2\,a^2 x^2}{3\,b\,y^3}} = \sqrt[6]{\frac{a^2 x^2}{4\,b^2 y^4}} \div \sqrt[6]{\frac{2\,a^2 x^2}{3\,b\,y^3}} = \sqrt[6]{\frac{a^2 x^2}{4\,b^2 y^4} \times \frac{3\,b\,y^3}{2\,a^2 x^2}}$

$= \sqrt[6]{\frac{3\,a^2\,b\,x^2\,y^3}{8\,a^2 b^2 x^2 y^4}} = \sqrt[6]{\frac{3}{8\,b\,y}}$

10. $\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} + \sqrt{3}} = \frac{2\sqrt{5} + 2\sqrt{3}}{\sqrt{5} + \sqrt{3}} = 2$

(11.)

$$\begin{array}{l|l}
a+2\sqrt{ab}+b-c & \sqrt{a}+\sqrt{b}-\sqrt{c} \\ \cline{2-2}
\underline{a+\sqrt{ab}-\sqrt{ac}} & \sqrt{a}+\sqrt{b}+\sqrt{c} \\
\qquad \sqrt{ab}+\sqrt{ac}+b & \\
\qquad \underline{\sqrt{ab} \qquad\quad +b-\sqrt{bc}} & \\
\qquad\qquad\quad \sqrt{ac} \qquad +\sqrt{bc}-c & \\
\qquad\qquad\quad \underline{\sqrt{ac} \qquad +\sqrt{bc}-c} &
\end{array}$$

12. $\sqrt[3]{a^5 b^2+a^3 b^4} \div \sqrt[3]{\frac{a^2-b^2}{8\,b}} = \sqrt[3]{(a^5 b^2+a^3 b^4)\left(\frac{8\,b}{a^2-b^2}\right)}$

$= \sqrt[3]{\frac{8\,a^3 b^3 (a^2+b^2)}{a^2-b^2}} = 2\,a\,b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$

POWERS OF RADICALS.

(Art. **246**, pp. 181, 182.)

2. $(9\,x\sqrt[3]{3\,a\,b^2})^2 = 81\,x^2\sqrt[3]{9\,a^2 b^4} = 81\,b\,x^2\sqrt[3]{9\,a^2 b}.$

3. $(\sqrt[4]{a\,b^2})^2 = \sqrt[4]{a^2 b^4} = b\sqrt[4]{a^2} = b\sqrt{a}.$

Or, $(\sqrt[4]{a\,b^2})^2 = \sqrt{a\,b^2} = b\sqrt{a}.$

4. $(\frac{1}{3}\sqrt{3})^3 = \frac{1}{27}\sqrt{27} = \frac{1}{27}\sqrt{9\times 3} = \frac{3}{27}\sqrt{3} = \frac{1}{9}\sqrt{3}.$

5. $(-\sqrt[3]{a^2})^4 = \sqrt[3]{a^8} = \sqrt[3]{a^6\times a^2} = a^2\sqrt[3]{a^2}.$

6. $(2+\sqrt{3})^2 = 4+4\sqrt{3}+3 = 7+4\sqrt{3}.$

7. $(a-b\,x^{\frac{1}{2}})^3 = a^3-3\,a^2 b x^{\frac{1}{2}}+3\,a b^2 x-b^3 x^{\frac{3}{2}}.$ (Art. 208.)

8. $(2\sqrt{a-b})^3 = 8\sqrt{(a-b)^3} = 8\,(a-b)\sqrt{a-b}.$

ROOTS OF RADICALS.

(ART. **247**, pp. 183, 184.)

1. $\sqrt[3]{\sqrt{a}} = \sqrt[6]{a}$.

2. $(64\sqrt[3]{7})^{\frac{1}{3}} = 64^{\frac{1}{3}}\sqrt[9]{7} = 4\sqrt[9]{7}$.

3. $(a^2\sqrt[3]{5})^{\frac{1}{2}} = a\sqrt[6]{5}$.

4. $(24\sqrt{3a})^{\frac{1}{2}} = (4\sqrt{108a})^{\frac{1}{2}} = 2\sqrt[4]{108a}$.

5. $[(x+y)\sqrt{x+y}]^{\frac{1}{3}} = (\sqrt{(x+y)^3})^{\frac{1}{3}} = \sqrt{x+y}$

6. $\left(\frac{a}{5}\sqrt{\frac{a}{5}}\right)^{\frac{1}{3}} = \left(\sqrt{\frac{a^3}{125}}\right)^{\frac{1}{3}} = \sqrt{\frac{a}{5}} = \sqrt{\frac{5a}{25}} = \frac{1}{5}\sqrt{5a}$.

7. $(2\sqrt{a^2-x})^{\frac{1}{4}} = (\sqrt{4(a^2-x)})^{\frac{1}{4}} = \sqrt[8]{4(a^2-x)}$.

(9.)

$$\begin{array}{r|l|l}
 & 9a+36(3ax)^{\frac{1}{2}}+108x & 3a^{\frac{1}{2}}+6(3x)^{\frac{1}{2}} \\
 & 9a & \\ \cline{2-2}
6a^{\frac{1}{2}}+6(3x)^{\frac{1}{2}} & 36(3ax)^{\frac{1}{2}}+108x & \\
 & 36(3ax)^{\frac{1}{2}}+108x & \\ \cline{2-2}
\end{array}$$

The square root may also be obtained by Art. 226.

RATIONALIZATION.

(ART. **251**, pp. 186–188.)

1. The coefficient, 3, is already rational, and $\sqrt{a^3}$ is rationalized by Art. 249.

$$3\sqrt{a^3}\times\sqrt{a} = 3a^{\frac{3}{2}}\times a^{\frac{1}{2}} = 3a^2.$$

2. $(a+\sqrt{b})(a-\sqrt{b}) = a^2-b$. (Art. 250.)

3. Let $n=4$ in Art. 250, 1; then

$$(x+y)(x^3-x^2y+xy^2-y^3)=x^4-y^4$$

Or, $(5^{\frac{1}{4}}+3^{\frac{1}{4}})(5^{\frac{3}{4}}-5^{\frac{1}{2}}\times 3^{\frac{1}{4}}+5^{\frac{1}{4}}\times 3^{\frac{1}{2}}-3^{\frac{3}{4}})=5-3=2$, in which $5^{\frac{3}{4}}-5^{\frac{1}{2}}\times 3^{\frac{1}{4}}+5^{\frac{1}{4}}\times 3^{\frac{1}{2}}-3^{\frac{3}{4}}$ is the rationalizing factor, and $5-3$, or 2, is the quantity after it is rationalized.

4. $\sqrt[5]{(a-b)^2}\times\sqrt[5]{(a-b)^3}=\sqrt[5]{(a-b)^5}=a-b.$

5. Let $n=6$ in Art. 250, 2; then

$$(x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)=x^6-y^6$$

Or, $(a^{\frac{1}{2}}-b^{\frac{2}{3}})(a^{\frac{5}{2}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}})=$ a^3-b^4.

7. $\dfrac{b}{\sqrt[3]{a}}\times\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}=\dfrac{b\sqrt[3]{a^2}}{a}.$

8. $\dfrac{b}{\sqrt[n]{a}}\times\dfrac{\sqrt[n]{a^{n-1}}}{\sqrt[n]{a^{n-1}}}=\dfrac{b\sqrt[n]{a^{n-1}}}{a}.$

9. $\dfrac{3}{\sqrt{5}-\sqrt{2}}\times\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\dfrac{3(\sqrt{5}+\sqrt{2})}{5-2}=\sqrt{5}+\sqrt{2}.$

10. $\dfrac{a}{\sqrt{b}+\sqrt{c}}\times\dfrac{\sqrt{b}-\sqrt{c}}{\sqrt{b}-\sqrt{c}}=\dfrac{a(\sqrt{b}-\sqrt{c})}{b-c}.$

11. $(\sqrt{8}-\sqrt{1}-\sqrt{3})(\sqrt{8}+\sqrt{1}-\sqrt{3})=10-2\sqrt{24}$

$$(10-2\sqrt{24})(10+2\sqrt{24})=100-96=4.$$

12. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\times\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=$

$$\frac{a+x-2\sqrt{a^2-x^2}+a-x}{a+x-a+x}=\frac{2a-2\sqrt{a^2-x^2}}{2x}=\frac{a-\sqrt{a^2-x^2}}{x}.$$

14. $\frac{\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3\sqrt{2}+2}{9-2} = \frac{4.2426+2}{7} = .8918.$

IMAGINARY QUANTITIES.

(Art. 256, pp. 190–192.)

1 and 2. The answers may be obtained directly by the rule found in Art. 255.

3. The imaginary terms cancel each other.

4. The fifth power of $\sqrt{-1}$ is the same as the first power. (See Rule, p. 191.)

$$(\sqrt{-a^2})^5 = (a\sqrt{-1})^5 = a^5\sqrt{-1}.$$

5. $(a+\sqrt{-b})\,(a-\sqrt{-b}) = a^2-(-b) = a^2+b.$

6. $\frac{\sqrt{-a}}{\sqrt{-b}} = \sqrt{\frac{-a}{-b}} = \sqrt{\frac{a}{b}}.$

7. $(4+\sqrt{-7})\,\sqrt{-2} = 4\sqrt{-2}-\sqrt{14}.$

8. $(3+4\sqrt{-1})^2 = 9+24\sqrt{-1}-16 = -7+24\sqrt{-1}.$

9. $\frac{1+\sqrt{-1}}{1-\sqrt{-1}} \times \frac{1+\sqrt{-1}}{1+\sqrt{-1}} = \frac{1+2\sqrt{-1}-1}{1-(-1)} = \frac{2\sqrt{-1}}{2}$
$= \sqrt{-1}.$

10. $\frac{4+\sqrt{-2}}{2-\sqrt{-2}} \times \frac{2+\sqrt{-2}}{2+\sqrt{-2}} = \frac{8+6\sqrt{-2}-2}{4+2}$
$= \frac{6+6\sqrt{-2}}{6} = 1+\sqrt{-2}.$

11. $(c \pm \sqrt{-d})^3 = c^3 \pm 3c^2\sqrt{-d} + 3c(\sqrt{-d})^2 \pm \sqrt{-d^3}$
$= c^3 \pm 3c^2\sqrt{-d} - 3cd \pm d\sqrt{-d}.$ (Art. 208.)

12. $\sqrt{-a^2}\times\sqrt{-b^2}\times\sqrt{-c^2}=a\sqrt{-1}\times b\sqrt{-1}\times c\sqrt{-1}$ $=abc\,(\sqrt{-1})^3=-abc\sqrt{-1}$.

13. $(\sqrt{-a^2})\,(\sqrt{-b^2})\,(\sqrt{-c^2})\,(\sqrt{-d^2})=(a\sqrt{-1})\,(b\sqrt{-1})$ $(c\sqrt{-1})\,(d\sqrt{-1})=abcd\,(\sqrt{-1})^4=abcd$.

14. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}+\dfrac{a-\sqrt{-b}}{a+\sqrt{-b}}=\dfrac{a^2+2a\sqrt{-b}-b}{a^2+b}+$ $\dfrac{a^2-2a\sqrt{-b}-b}{a^2+b}=\dfrac{2\,(a^2-b)}{a^2+b}$

PROPERTIES OF QUADRATIC SURDS.

(Art. **265**, pp. 194–196.)

2. Here $a=11$, $b=72$, and $c=\sqrt{a^2-b}=\sqrt{121-72}$ $=\sqrt{49}=7$. Hence, from (7),

$$\sqrt{11+\sqrt{72}}=\sqrt{\frac{11+7}{2}}+\sqrt{\frac{11-7}{2}}=3+\sqrt{2}.$$

3. Here $a=6$, $b=20$, and $c=\sqrt{36-20}=\sqrt{16}=4$. Hence, from (7),

$$\sqrt{6+\sqrt{20}}=\sqrt{\frac{6+4}{2}}+\sqrt{\frac{6-4}{2}}=\sqrt{5}+1.$$

4. Here $a=1$, $\sqrt{b}=4\sqrt{-3}=\sqrt{-48}$, or $b=-48$, and $c=\sqrt{1+48}=\sqrt{49}=7$. Hence, from (7),

$$\sqrt{1+4\sqrt{-3}}=\sqrt{\frac{1+7}{2}}+\sqrt{\frac{1-7}{2}}=2+\sqrt{-3}.$$

5. Here $a = 43$, $\sqrt{b} = 15\sqrt{8} = \sqrt{1800}$, or $b = 1800$, and $c = \sqrt{1849 - 1800} = \sqrt{49} = 7$. Hence, from (8),

$$\sqrt{43 - 15\sqrt{8}} = \sqrt{\frac{43+7}{2}} - \sqrt{\frac{43-7}{2}} = 5 - \sqrt{18} = 5 - 3\sqrt{2}.$$

6. Here $a = 12$, $\sqrt{b} = 2\sqrt{35} = \sqrt{140}$, or $b = 140$, and $c = \sqrt{144 - 140} = \sqrt{4} = 2$. Hence, from (7),

$$\sqrt{12 + 2\sqrt{35}} = \sqrt{\frac{12+2}{2}} + \sqrt{\frac{12-2}{2}} = \sqrt{7} + \sqrt{5}.$$

7. Here $a = 2m$, $\sqrt{b} = 2\sqrt{m^2 - n^2} = \sqrt{4(m^2 - n^2)}$, or $b = 4m^2 - 4n^2$, and $c = \sqrt{4m^2 - 4m^2 + 4n^2} = \sqrt{4n^2} = 2n$. Hence, from (7),

$$\sqrt{2m + 2\sqrt{m^2 - n^2}} = \sqrt{\frac{2m+2n}{2}} + \sqrt{\frac{2m-2n}{2}} = \sqrt{m+n}$$
$$+ \sqrt{m-n}.$$

8. Here $a = 0$, $\sqrt{b} = 2\sqrt{-1} = \sqrt{-4}$, or $b = -4$, and $c = \sqrt{0 + 4} = 2$. Hence, from (8),

$$\sqrt{-2\sqrt{-1}} = \sqrt{\frac{0+2}{2}} - \sqrt{\frac{0-2}{2}} = 1 - \sqrt{-1}$$

9. Here $a = \frac{a^2}{4}$, $\sqrt{b} = \frac{b}{2}\sqrt{a^2 - b^2} = \sqrt{\frac{a^2b^2 - b^4}{4}}$, or $b = \frac{a^2b^2 - b^4}{4}$, and $c = \sqrt{\frac{a^4}{16} - \frac{a^2b^2 - b^4}{4}} = \sqrt{\frac{a^4 - 4a^2b^2 + 4b^4}{16}} = \frac{a^2 - 2b^2}{4}$.
Hence, from (7),

$$\sqrt{\frac{a^2}{4} + \frac{b}{2}\sqrt{a^2 - b^2}} = \sqrt{\frac{\frac{a^2}{4} + \frac{a^2 - 2b^2}{4}}{2}} + \sqrt{\frac{\frac{a^2}{4} - \frac{a^2 - 2b^2}{4}}{2}}$$
$$= \sqrt{\frac{2a^2 - 2b^2}{8}} + \sqrt{\frac{2b^2}{8}} = \sqrt{\tfrac{1}{4}(a^2 - b^2)} + \sqrt{\frac{b^2}{4}}$$
$$= \tfrac{1}{2}\sqrt{a^2 - b^2} + \frac{b}{2}.$$

RADICAL EQUATIONS.

(Art. **267**, pp. 196–199.)

6. Given $\sqrt{x} + 6 = \sqrt{12 + x}$

Squaring, $x + 12\sqrt{x} + 36 = 12 + x$

Transposing and uniting, $12\sqrt{x} = -24$

Dividing by 12, $\sqrt{x} = -2$

Squaring, $x = 4$,

in which 4, the value of x, is limited to the negative square root.

7. Given $\sqrt{x} + \sqrt{x - 7} = \frac{21}{\sqrt{x - 7}}$

Clearing of fractions, $\sqrt{x^2 - 7x} + x - 7 = 21$

Transposing and uniting, $\sqrt{x^2 - 7x} = 28 - x$

Squaring, $x^2 - 7x = 784 - 56x + x^2$

Transposing and uniting, $49x = 784$

Whence, $x = 16$

8. Given $\sqrt[3]{3x + 4} + 3 = 6$

Or, $\sqrt[3]{3x + 4} = 3$

Cubing, $3x + 4 = 27$

Or, $3x = 23$

Whence, $x = 7\frac{2}{3}$

9. Given $\frac{x}{\sqrt{b^2 + x} - b} = c$

Clearing of fractions, $x = c\sqrt{b^2 + x} - bc$

Transposing, $x + bc = c\sqrt{b^2 + x}$

Squaring, $x^2 + 2bcx + b^2c^2 = b^2c^2 + c^2x$

Or, $x^2 = c^2x - 2bcx$

Dividing by x, $x = c^2 - 2bc$

10. Given $(x+2)^{\frac{1}{m}}=(x^2+10x+1)^{\frac{1}{2m}}$

Raising to 2*m*th power, $(x+2)^2=x^2+10x+1$

Or, $x^2+4x+4=x^2+10x+1$

Whence, $6x=3$

Dividing by 6, $x=\frac{3}{6}=\frac{1}{2}$

11. Given $\sqrt{x-32}=16-\sqrt{x}$

Squaring, $x-32=256-32\sqrt{x}+x$

Transposing and uniting, $32\sqrt{x}=288$

Dividing by 32, $\sqrt{x}=9$

Squaring, $x=81$

12. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$

Squaring, $\frac{b}{a+x}+2\sqrt{\frac{bc}{a^2-x^2}}+\frac{c}{a-x}=\sqrt{\frac{4bc}{a^2-x^2}}$

Dropping equal terms, $\frac{b}{a+x}+\frac{c}{a-x}=0$

Clearing of fractions, $ab-bx+ac+cx=0$

Or, $(b-c)x=a(b+c)$

Whence, $x=a\left(\frac{b+c}{b-c}\right)$

13. Given $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$

Squaring, $a^2+2ax+x^2=a^2+x\sqrt{b^2+x^2}$

Dividing by x, etc., $2a+x=\sqrt{b^2+x^2}$

Squaring, $4a^2+4ax+x^2=b^2+x^2$

Or, $4ax=b^2-4a^2$

Whence, $x=\frac{b^2}{4a}-a$

14. Given $(x^{\frac{1}{2}}+3a^2)^{\frac{1}{2}}-(x^{\frac{1}{2}}-3a^2)^{\frac{1}{2}}=\frac{2a^{\frac{1}{2}}x^{\frac{1}{4}}}{b^{\frac{1}{2}}}$

Squaring, $x^{\frac{1}{2}}+3a^2-2(x-9a^4)^{\frac{1}{2}}+x^{\frac{1}{2}}-3a^2=\frac{4ax^{\frac{1}{2}}}{b}$

Clearing of fractions, etc., $2bx^{\frac{1}{2}}-2b(x-9a^4)^{\frac{1}{2}}=4ax^{\frac{1}{2}}$

Transposing, etc., $bx^{\frac{1}{2}}-2ax^{\frac{1}{2}}=b(x-9a^4)^{\frac{1}{2}}$

Squaring, $b^2x-4abx+4a^2x=b^2x-9a^4b^2$

Dividing by a, etc., $-4bx+4ax=-9a^3b^2$

Factoring, etc., $4(b-a)x=9a^3b^2$

Whence, $x=\frac{9a^3b^2}{4(b-a)}$

QUADRATIC EQUATIONS.

PURE QUADRATIC EQUATIONS.

(ARTS. 271, 272, pp. 200–203.)

2. Given $4x^2-7=29$

Transposing and uniting, $4x^2=36$

Dividing by 4, $x^2=9$

Evolving, $x=\pm 3$

3. Given $5x^2+5=3x^2+55$

Transposing and uniting, $2x^2=50$

Dividing by 2, $x^2=25$

Evolving, $x=\pm 5$

4. Given $7x^2-5=3x^2+11$

Transposing and uniting, $4x^2=16$

Dividing by 4, $x^2=4$

Evolving, $x=\pm 2$

5. Given $\frac{5}{4+x}+\frac{5}{4-x}=\frac{8}{3}$

Clearing of fractions, $60-15x+60+15x=128-8x^2$

Or, $8x^2=8$

Dividing by 8, $x^2=1$

Evolving, $x=\pm 1$

6. Given $x^2-ab=d$

Transposing, $x^2=d+ab$

Evolving, $x=\pm\sqrt{d+ab}$

7. Given $ax^2+n=m-c$

Transposing, $ax^2=m-c-n$

Dividing by a, $x^2=\frac{m-c-n}{a}$

Evolving, $x=\pm\sqrt{\frac{m-c-n}{a}}$

8. Given $\frac{x^2}{2}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{335}{24}$

Clearing of fractions, $12x^2-72+10x^2=7-24x^2+335$

Transposing and uniting, $46x^2=414$

Dividing by 46, $x^2=9$

Evolving, $x=\pm 3$

9. Given $13-\sqrt{3x^2+16}=5$

Or, $\sqrt{3x^2+16}=8$

Squaring, $3x^2+16=64$

Transposing and uniting, $3x^2=48$

Dividing by 3, $x^2=16$

Evolving, $x=\pm 4$

10. Given $x+\sqrt{x^2+a}=\frac{2a}{\sqrt{x^2+a}}$

Clearing of fractions, $x\sqrt{x^2+a}+x^2+a=2a$

Transposing and uniting, $x\sqrt{x^2+a}=a-x^2$
Squaring, $x^4+ax^2=a^2-2ax^2+x^4$
Transposing and uniting, $3ax^2=a^2$
Dividing by $3a$, $x^2=\frac{a}{3}$
Evolving, $x=\pm\sqrt{\frac{a}{3}}=\pm\frac{1}{3}\sqrt{3a}$

11. Given $\frac{1}{1-\sqrt{1-x^2}}-\frac{1}{1+\sqrt{1-x^2}}=\frac{\sqrt{3}}{x^2}$

Clear. of fract., $1+\sqrt{1-x^2}-1+\sqrt{1-x^2}=\sqrt{3}$
Uniting, $2\sqrt{1-x^2}=\sqrt{3}$
Squaring, $4-4x^2=3$
Whence, $4x^2=1$
Dividing by 4, $x^2=\frac{1}{4}$
Evolving, $x=\pm\frac{1}{2}$

12. Given $\frac{\sqrt{a+x}}{\sqrt{x}}+\frac{\sqrt{a-x}}{\sqrt{x}}=\sqrt{\frac{x}{6}}$

Mult. by $\sqrt{x}$, $\sqrt{a+x}+\sqrt{a-x}=\frac{x}{\sqrt{6}}$
Squaring, $a+x+2\sqrt{a^2-x^2}+a-x=\frac{x^2}{6}$
Clearing of fractions, etc., $12\sqrt{a^2-x^2}=x^2-12a$
Squaring, $144a^2-144x^2=x^4-24ax^2+144a^2$
Whence, $x^4=24ax^2-144x^2$
Dividing by x^2, $x^2=24a-144$
Evolving, $x=\pm\sqrt{24a-144}$

14. Given
$$\begin{cases} 3x+3y=5x & (1) \\ \underline{(x+y)x=960} & (2) \end{cases}$$

From (1), $y=\frac{2x}{3}$ (3)

Substituting in (2), $x^2+\frac{2x^2}{3}=960$

Clearing of fractions, etc., $5x^2=2880$

Dividing by 5, $x^2=576$

Evolving, $x=\pm 24$

Substituting in (3), $y=\pm\frac{2\times 24}{3}=\pm 16$

15. Given $\begin{cases} 3x^2+4y^2=76 & (1) \\ 3y^2-11x^2=4 & (2) \end{cases}$

Multiplying (1) by 3, $9x^2+12y^2=228 \quad (3)$

Multiplying (2) by 4, $-44x^2+12y^2=16 \quad (4)$

Subtracting (4) from (3), $53x^2=212 \quad (5)$

Dividing by 53, $x^2=4$

Evolving, $x=\pm 2$

Substituting in (1), $12+4y^2=76$

Transposing and uniting, $4y^2=64$

Dividing by 4, $y^2=16$

Evolving, $y=\pm 4$

16. Given $\begin{cases} x^2-xy=54 & (1) \\ xy-y^2=18 & (2) \end{cases}$

Dividing (1) by (2), $\frac{x(x-y)}{y(x-y)}=\frac{x}{y}=3 \quad (3)$

Or, $x=3y \quad (4)$

Substituting in (2), $3y^2-y^2=18$

Or, $2y^2=18$

Dividing by 2, $y^2=9$

Evolving, $y=\pm 3$

Substituting in (4), $x=\pm 9$

17. Given $\begin{cases} x^2+xy=60 & (1) \\ xy+y^2=84 & (2) \end{cases}$

Adding (1) and (2), $x^2+2xy+y^2=144 \quad (3)$

Evolving, $x+y=\pm 12$ (4)

Dividing (1) by (4), $x=\pm 5$

Dividing (2) by (4), $y=\pm 7$

18. Given $\begin{cases} x^2-y^2=2268 & (1) \\ \dfrac{63x^2}{4y}=28y & (2) \end{cases}$

Multiplying (2) by $\frac{4y}{7}$, $9x^2-16y^2=0$ (3)

Multiplying (1) by 9, $9x^2-9y^2=20412$ (4)

Subtracting (3) from (4), $7y^2=20412$

Dividing by 7, $y^2=2916$

Evolving, $y=\pm 54$

Substituting in (1), $x^2-2916=2268$

Transposing and uniting, $x^2=5184$

Evolving, $x=\pm 72$

PROBLEMS

LEADING TO PURE EQUATIONS.

(Pp. 203-205.)

2. Let $x=$ side of each lot,

and $3x^2=$ contents of the three.

Then, $3x^2+193=25^2$

Transposing and uniting, $3x^2=432$

Dividing by 3, $x^2=144$

Evolving, $x=12$, side of each.

3. Let $x=$ A's capital,

and $y=$ B's capital.

Then $\frac{xy}{100}=$ A's gain,

and $\frac{y^2}{200}=$ B's gain.

By the conditions, $x + \frac{xy}{100} = 27$ (1)

and $\frac{y^2}{200} = 32$ (2)

From (2), $y^2 = 6400$ (3)

Evolving, $y = 80$, B's capital.

Substituting in (1), $x + \frac{80x}{100} = 27$

Multiplying by 5, $5x + 4x = 135$

Uniting terms, $9x = 135$

Dividing by 9, $x = 15$, A's capital.

4. Let $5x =$ side of larger,

and $3x =$ side of smaller.

Then $25x^2 =$ contents of larger,

and $9x^2 =$ contents of smaller.

Therefore, $25x^2 - 9x^2 = 25600$

Or, $16x^2 = 25600$

Evolving, $4x = 160$

Dividing by 4, $x = 40$

Squaring, $x^2 = 1600$

Whence, $25x^2 = 40000$, larger.

Also, $9x^2 = 14400$, smaller.

6. Let $x =$ A's investment,

and $y =$ B's investment.

Then $\frac{xy}{100} =$ A's gain,

and $\frac{y^2}{400} =$ B's gain.

By the conditions, $x + \frac{xy}{100} = 11$ (1)

and $\frac{y^2}{400} = 36$ (2)

From (2), $y^2 = 14400$

Evolving, $y = 120$, B's investment.

Substituting in (1), $x + \frac{6x}{5} = 11$

Clearing of fractions, $5x + 6x = 55$

Uniting terms, $11x = 55$

Dividing by 11, $x = 5$, A's investment.

7. Let $x =$ the greater,

and $y =$ the less.

Then $(x - y)y = 42 \qquad (1)$

$(x - y)(x + y) = 133 \qquad (2)$

Dividing (2) by (1), $\frac{x+y}{y} = \frac{133}{42} = \frac{19}{6} \qquad (3)$

Clearing of fractions, $6x + 6y = 19y \qquad (4)$

Or, $6x = 13y \qquad (5)$

Whence, $x = \frac{13y}{6} \qquad (6)$

Substituting in (1), $\frac{13y^2}{6} - y^2 = 42$

Clearing of fractions, $13y^2 - 6y^2 = 252$

Or, $7y^2 = 252$

Whence, $y^2 = 36$

Evolving, $y = \pm 6$, the less.

Substituting in (6), $x = \pm 13$, the greater.

8. Let $x =$ no. weeks engaged.

Then $\frac{75}{x-4} =$ A's rate per week,

and $\frac{48}{x-7} =$ B's rate per week.

By the conditions, $\frac{75(x-7)}{x-4} = \frac{48(x-4)}{x-7}$

Clearing of fractions, $75(x-7)^2 = 48(x-4)^2$

Or, $25(x-7)^2 = 16(x-4)^2$

Evolving, $5(x-7) = 4(x-4)$
Or, $5x-35 = 4x-16$
Transposing and uniting, $x = 19$, whole no. weeks.
Hence, $x-4 = 15$, no. weeks A worked.
And, $x-7 = 12$, no. weeks B worked.
Also, $\frac{75}{x-4} = 5$, A's rate per week.
And, $\frac{48}{x-7} = 4$, B's rate per week.

AFFECTED QUADRATIC EQUATIONS.

FIRST METHOD OF COMPLETING THE SQUARE.

(Arts. 276, 277, pp. 206–210.)

4. Given $x^2-6x+12 = 4$
Transposing, $x^2-6x = -8$
Completing the square, $x^2-6x+9 = 1$
Evolving, $x-3 = \pm 1$
Whence, $x = 4$, or 2

5. Given $x^2-8x+50 = 98$
Transposing, $x^2-8x = 48$
Completing the square, $x^2-8x+16 = 64$
Evolving, $x-4 = \pm 8$
Whence, $x = 12$, or -4

6. Given $3x^2-3x+6 = 5\frac{1}{3}$
Transposing and dividing by 3, $x^2-x = -\frac{2}{9}$
Completing the square, $x^2-x+\frac{1}{4} = \frac{1}{36}$
Evolving, $x-\frac{1}{2} = \pm\frac{1}{6}$
Whence, $x = \frac{4}{6}$, or $\frac{2}{6} = \frac{2}{3}$, or $\frac{1}{3}$

7. Given $-3x^2+36x-105=0$
Transposing and div. by -3, $x^2-12x=-35$
Completing the square, $x^2-12x+36=1$
Evolving, $x-6=\pm 1$
Whence, $x=7$, or 5

8. Given $4x-\frac{36-x}{x}=46$

Clearing of fractions, $4x^2-36+x=46x$
Transposing and uniting, $4x^2-45x=36$
Dividing by 4, $x^2-\frac{45}{4}x=\frac{36}{4}$
Completing the square, $x^2-\frac{45}{4}x+\frac{2025}{64}=\frac{2601}{64}$
Evolving, $x-\frac{45}{8}=\pm\frac{51}{8}$
Whence, $x=\frac{96}{8}$, or $-\frac{6}{8}$, $=12$, or $-\frac{3}{4}$

9. Given $6x-30=3x^2$
Transposing, etc., $3x^2-6x=-30$
Dividing by 3, $x^2-2x=-10$
Completing the square, $x^2-2x+1=-9$
Evolving, $x-1=\pm\sqrt{-9}=\pm 3\sqrt{-1}$
Whence, $x=1\pm 3\sqrt{-1}$

10. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$

Transposing, $\frac{x^2}{2}-\frac{x}{3}=22\frac{1}{6}$

Multiplying by 2, $x^2-\frac{2x}{3}=44\frac{1}{3}$

Completing the square, $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{400}{9}$

Evolving, $x-\frac{1}{3}=\pm\frac{20}{3}$
Whence, $x=7$, or $-6\frac{1}{3}$

11. Given $x^2+ax=b$

Completing the square, $x^2+ax+\frac{a^2}{4}=b+\frac{a^2}{4}$

Evolving, $x+\frac{a}{2}=\pm\sqrt{b+\frac{a^2}{4}}$

Whence, $x=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$

13. The given equation reduces to $x^2+4x=45$, in which $\frac{p}{2}=2$, and $q=45$; hence,

$$x=-2\pm\sqrt{45+4}=-2\pm7=5, \text{ or } -9.$$

14. The given equation reduces to $x^2-x=\frac{3}{4}$, in which $\frac{p}{2}=-\frac{1}{2}$ and $q=\frac{3}{4}$; hence,

$$x=\tfrac{1}{2}\pm\sqrt{\tfrac{3}{4}+\tfrac{1}{4}}=\tfrac{1}{2}\pm\tfrac{2}{2}=\tfrac{3}{2}, \text{ or } -\tfrac{1}{2}.$$

15. The given equation reduces to $x^2+6x=-8$, in which $\frac{p}{2}=3$ and $q=-8$; hence,

$$x=-3\pm\sqrt{-8+9}=-3\pm1=-2, \text{ or } -4.$$

16. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$

Uniting terms, $3x-\frac{3x-3}{x-3}=\frac{3x-6}{2}$

Clearing of fractions, $6x^2-18x-6x+6=3x^2-15x+18$

Transposing and uniting terms, $3x^2-9x=12$

Dividing by 3, $x^2-3x=4$

Hence, $x=\frac{3}{2}\pm\sqrt{4+\frac{9}{4}}$

Or, $x=\frac{3}{2}\pm\frac{5}{2}$

Whence, $x=4, \text{ or } -1$

SECOND METHOD OF COMPLETING THE SQUARE.

(ART. 278, pp. 210–213.)

2. Given $3x^2 = 42 - 5x$

Or, $3x^2 + 5x = 42$

Multiplying by 12, $36x^2 + 60x = 504$

Adding 5^2, $36x^2 + 60x + 25 = 529$

Evolving, $6x + 5 = \pm 23$

Whence, $6x = 18$, or -28

Dividing by 6, $x = 3$, or $-4\frac{2}{3}$

3. Given $\sqrt{(4+x)(5-x)} = 2x - 10$

Squaring, $20 + x - x^2 = 4x^2 - 40x + 100$

Transposing, uniting, etc., $5x^2 - 41x = -80$

Multiplying by 20, $100x^2 - 820x = -1600$

Adding 41^2, $100x^2 - 820x + 1681 = 81$

Evolving, $10x - 41 = \pm 9$

Whence, $10x = 50$, or 32

Dividing by 10, $x = 5$, or $3\frac{1}{5}$

4. Given $\frac{6}{x+1} + \frac{2}{x} = 3$

Clearing of fractions, $6x + 2x + 2 = 3x^2 + 3x$

Transposing, uniting, etc., $3x^2 - 5x = 2$

Multiplying by 12, $36x^2 - 60x = 24$

Adding 5^2, $36x^2 - 60x + 25 = 49$

Evolving, $6x - 5 = \pm 7$

Whence, $6x = 12$, or -2

Dividing by 6, $x = 2$, or $-\frac{1}{3}$

5. Given $2x^2 - 7x + 3 = 0$

Multiplying by 8, $16x^2 - 56x = -24$

Adding 7^2, $16x^2-56x+49=25$
Evolving, $4x-7=\pm 5$
Whence, $4x=12$, or 2
Dividing by 4, $x=3$, or $\frac{1}{2}$

6. Given $acx^2-bcx+adx=bd$
Mult. by $4ac$, $4a^2c^2x^2-4ac(bc-ad)x=4abcd$
Adding $(bc-ad)^2$,
$4a^2c^2x^2-4ac(bc-ad)x+(bc-ad)^2=b^2c^2+2abcd+a^2d^2$
Evolving, $2acx-(bc-ad)=\pm(bc+ad)$
Transposing and uniting, $2acx=2bc$, or $-2ad$
Dividing by $2ac$, $x=\frac{b}{a}$, or $-\frac{d}{c}$

7. Given $110x^2-21x+1=0$
Multiplying by 440, $48400x^2-9240x=-440$
Adding 21^2, $48400x^2-9240x+441=1$
Evolving, $220x-21=\pm 1$
Whence, $220x=22$, or 20
Dividing by 220, $x=\frac{1}{10}$, or $\frac{1}{11}$

8. Given $125x^2-7x=17\frac{1}{5}$
Clearing of fractions, $625x^2-35x=86$
Multiplying by 2500, $1562500x^2-87500x=215000$
Adding 35^2, $1562500x^2-87500x+1225=216225$
Evolving, $1250x-35=\pm 465$
Whence, $1250x=500$, or -430
Dividing by 1250, $x=\frac{2}{5}$, or $-\frac{43}{125}$

9. Given $ax^2-2bx=c$
Multiplying by a, $a^2x^2-2abx=ac$

Adding b^2, $a^2x^2-2abx+b^2=ac+b^2$

Evolving, $ax-b=\pm\sqrt{ac+b^2}$

Whence, $ax=b\pm\sqrt{ac+b^2}$

Dividing by a, $x=\frac{b}{a}\pm\frac{1}{a}\sqrt{ac+b^2}$

THIRD METHOD OF COMPLETING THE SQUARE.

(ART. 279, pp. 213, 214.)

1. Given $4x^2+16x=33$

Adding 16, $4x^2+16x+16=49$

Evolving, $2x+4=\pm 7$

Whence, $2x=3$, or -11

Dividing by 2, $x=\frac{3}{2}$, or $-\frac{11}{2}$

2. Given $3x^2+2x+6=11$

Multiplying by 3, etc., $9x^2+6x=15$

Adding $\left(\frac{6x}{2\sqrt{9x^2}}\right)^2$, $9x^2+6x+1=16$

Evolving, $3x+1=\pm 4$

Whence, $3x=3$, or -5

Dividing by 3, $x=1$, or $-\frac{5}{3}$

3. Given $16x^2+15x=34$

Adding $\left(\frac{15}{2\sqrt{16}}\right)^2$, $16x^2+15x+\frac{225}{64}=\frac{2401}{64}$

Evolving, $4x+\frac{15}{8}=\pm\frac{49}{8}$

Whence, $4x=\frac{34}{8}$, or $-\frac{64}{8}=\frac{17}{4}$, or -8

Dividing by 4, $x=\frac{17}{16}$, or -2

4. Given $9x^{-2}-12x^{-1}=-3$

Adding $\left(\frac{12}{2\sqrt{9}}\right)^2$, $9x^{-2}-12x^{-1}+4=1$

Evolving, $3x^{-1}-2=\pm 1$

Whence, $3x^{-1}=3, \text{ or } 1$

Dividing by 3, $x^{-1}=1, \text{ or } \frac{1}{3}$

Hence, $x=1, \text{ or } 3$

5. Given $\frac{(2x-3)(2x-3)}{x}=8$

Clearing of fractions, etc., $4x^2-12x+9=8x$

Or, $4x^2-20x=-9$

Adding $\left(\frac{20}{2\sqrt{4}}\right)^2$, $4x^2-20x+25=16$

Evolving, $2x-5=\pm 4$

Whence, $2x=9, \text{ or } 1$

Dividing by 2, $x=\frac{9}{2}, \text{ or } \frac{1}{2}$

6. Given $\frac{9x^2}{16}+\frac{30x}{8}=-4$

Adding $(\frac{30}{8}\div 2\sqrt{\frac{9}{16}})^2$, $\frac{9x^2}{16}+\frac{30x}{8}+\frac{25}{4}=\frac{9}{4}$

Evolving, $\frac{3x}{4}+\frac{5}{2}=\pm\frac{3}{2}$

Whence, $\frac{3x}{4}=-1, \text{ or } -4$

Clearing of fractions, $3x=-4, \text{ or } -16$

Dividing by 3, $x=-1\frac{1}{3}, \text{ or } -5\frac{1}{3}$

7. Given $\frac{4x^2}{49}+\frac{16x}{42}=\frac{20}{3}$

Adding $(\frac{8}{21}\div 2\sqrt{\frac{4}{49}})^2$, $\frac{4x^2}{49}+\frac{8x}{21}+\frac{4}{9}=\frac{64}{9}$

Evolving, $\frac{2x}{7}+\frac{2}{3}=\pm\frac{8}{3}$

Whence, $\frac{2x}{7}=2, \text{ or } -\frac{10}{3}$

Dividing by $\frac{2}{7}$, $x=7, \text{ or } -11\frac{2}{3}$

PROMISCUOUS EXAMPLES.

(ART. 280, pp. 214–217.)

1. Given $x^2 - x + 3 = 45$

Transposing and compl. square, $x^2 - x + \frac{1}{4} = \frac{169}{4}$

Evolving, $x - \frac{1}{2} = \pm \frac{13}{2}$

Whence, $x = 7$, or -6

2. Given $(x - 12)(x + 2) = 0$

Developing, $x^2 - 10x - 24 = 0$

Transposing and compl. square, $x^2 - 10x + 25 = 49$

Evolving, $x - 5 = \pm 7$

Whence, $x = 12$, or -2

3. Given $2x^2 - x = 21$

Multiplying by 8, $16x^2 - 8x = 168$

Completing the square, $16x^2 - 8x + 1 = 169$

Evolving, $4x - 1 = \pm 13$

Whence, $4x = 14$, or -12

Dividing by 4, $x = \frac{7}{2}$, or -3

4. Given $4x - \frac{14 - x}{x + 1} = 14$

Clearing of fractions, $4x^2 + 4x - 14 + x = 14x + 14$

Or, $4x^2 - 9x = 28$

Adding $\left(\frac{9}{2\sqrt{4}}\right)^2$, $4x^2 - 9x + \frac{81}{16} = \frac{529}{16}$

Evolving, $2x - \frac{9}{4} = \pm \frac{23}{4}$

Whence, $2x = 8$, or $-\frac{7}{2}$

Dividing by 2, $x = 4$, or $-\frac{7}{4}$

6. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$

Clearing of fractions. $90x - 126 + 18x = 22x^2$

Or, $22x^2 - 108x = -126$

Dividing by 22, $x^2 - \frac{54}{11}x = -\frac{63}{11}$

Completing the square, $x^2 - \frac{54}{11}x + \frac{729}{121} = \frac{36}{121}$

Evolving, $x - \frac{27}{11} = \pm \frac{6}{11}$

Whence, $x = 3$, or $\frac{21}{11}$

7. Given $x + \sqrt{5x + 10} = 8$

Transposing, $\sqrt{5x + 10} = 8 - x$

Squaring, $5x + 10 = 64 - 16x + x^2$

Or, $x^2 - 21x = -54$

Multiplying by 4, $4x^2 - 84x = -216$

Completing the square, $4x^2 - 84x + 441 = 225$

Evolving, $2x - 21 = \pm 15$

Whence, $2x = 36$, or 6

Dividing by 2, $x = 18$, or 3

8. Given $x + \sqrt{10x + 6} = 9$

Transposing, $\sqrt{10x + 6} = 9 - x$

Squaring, $10x + 6 = 81 - 18x + x^2$

Or, $x^2 - 28x = -75$

Completing the square, $x^2 - 28x + 196 = 121$

Evolving, $x - 14 = \pm 11$

Whence, $x = 25$, or 3

9. Given $48x^{-2} + 32x^{-1} = 11$

Multiplying by 3, $144x^{-2} + 96x^{-1} = 33$

Adding $\left(\frac{96}{2\sqrt{144}}\right)^2$, $144x^{-2} + 96x^{-1} + 16 = 49$

Evolving, $12x^{-1} + 4 = \pm 7$

Whence, $12x^{-1} = 3$, or -11

Dividing by 12, $x^{-1} = \frac{1}{4}$, or $-\frac{11}{12}$

Hence, $x = 4$, or $-1\frac{1}{11}$

10. Given $81x^2 - 36x + 4 = 49$

Or, $81x^2 - 36x = 45$

Adding $\left(\frac{36}{2\sqrt{81}}\right)^2$, $81x^2 - 36x + 4 = 49$

Evolving, $9x - 2 = \pm 7$

Whence, $9x = 9$, or -5

Dividing by 9, $x = 1$, or $-\frac{5}{9}$

11. Given $\sqrt{3x - 5} = \frac{\sqrt{7x^2 + 36x}}{x}$

Squaring, $3x - 5 = \frac{7x^2 + 36x}{x^2}$

Multiplying by x, $3x^2 - 5x = 7x + 36$

Or, $3x^2 - 12x = 36$

Dividing by 3, $x^2 - 4x = 12$

Completing the square, $x^2 - 4x + 4 = 16$

Evolving, $x - 2 = \pm 4$

Whence, $x = 6$, or -2

12. Given $\frac{x^2}{5} + 20x = 3x^2 - 80$

Clearing of fractions, $x^2 + 100x = 15x^2 - 400$

Or, $14x^2 - 100x = 400$

Dividing by 2, $7x^2 - 50x = 200$

Multiplying by 7, $49x^2 - 350x = 1400$

Completing the square, $49x^2 - 350x + 625 = 2025$

Evolving, $7x - 25 = \pm 45$

Whence, $7x = 70$, or -20

Dividing by 7, $x = 10$, or $-2\frac{6}{7}$

13. Given $\frac{x - 1}{x + 1} + \frac{x + 3}{x - 3} = 2\left(\frac{x + 2}{x - 2}\right)$

Reducing the first member to a common denominator,

$$\frac{x^2-4x+3+x^2+4x+3}{x^2-2x-3}=2\left(\frac{x+2}{x-2}\right)$$

Or, $\dfrac{x^2+3}{x^2-2x-3}=\dfrac{x+2}{x-2}$

Clearing of fractions, $x^3-2x^2+3x-6=x^3-7x-6$

Or, $2x^2-10x=0$

Dividing by 2, $x^2-5x=0$

Dividing by x, $x-5=0$

Or, $x=5$

If the square be completed in the usual way, we have

$$4x^2-20x+25=25$$

Evolving, $2x-5=\pm 5$

Whence, $2x=10$, or 0

Dividing by 2, $x=5$, or 0

14. Given $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{1}{3}+\dfrac{3}{x}$

Clearing of fractions, $3x^2+12=2x+18$

Or, $3x^2-2x=6$

Multiplying by 3, $9x^2-6x=18$

Completing the square, $9x^2-6x+1=19$

Evolving, $3x-1=\pm\sqrt{19}$

Whence, $3x=1\pm\sqrt{19}$

Dividing by 3, $x=\dfrac{1\pm\sqrt{19}}{3}$

15. Given $\dfrac{21}{5-x}-\dfrac{x}{7}=3\frac{2}{7}$

Clearing of fractions, $147-5x+x^2=115-23x$

Or, $x^2+18x=-32$

Completing the square, $x^2+18x+81=49$

Evolving, $x+9=\pm 7$

Whence, $x=-2$, or -16

16. Given $x^2 - 2ax + a^2 - b^2 = 0$

Completing the square, $x^2 - 2ax + a^2 = b^2$

Evolving, $x - a = \pm b$

Whence, $x = a \pm b$

17. Given $6x - 33 = 3x^2$

Transposing, dividing by 3, etc., $x^2 - 2x = -11$

Completing the square, $x^2 - 2x + 1 = -10$

Evolving, $x - 1 = \pm\sqrt{-10}$

Whence, $x = 1 \pm \sqrt{-10}$

18. Given $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$

Clearing of fractions,

$$abx = abx + b^2x + bx^2 + a^2x + abx + ax^2 + a^2b + ab^2 + abx$$

Or, $(a+b)x^2 + (a^2 + 2ab + b^2)x = -ab(a+b)$

Dividing by $a+b$, $x^2 + (a+b)x = -ab$

Compl. square, $x^2 + (a+b)x + \left(\frac{a+b}{2}\right)^2 = \frac{a^2 - 2ab + b^2}{4}$.

Evolving, $x + \frac{a+b}{2} = \pm \frac{a-b}{2}$

Whence, $x = \frac{-a - b \pm (a-b)}{2}$

Or, $x = -b$, or $-a$

The first part of the above operation may be abbreviated a little, as follows:

Transposing, $\frac{1}{a+b+x} - \frac{1}{x} = \frac{1}{a} + \frac{1}{b}$

Reducing the fractions in each member to a common denominator,

$$\frac{-(a+b)}{(a+b+x)x} = \frac{a+b}{ab}$$

Dividing by $a+b$, $\frac{-1}{(a+b+x)x} = \frac{1}{ab}$

Whence, $x^2 + (a+b)x = -ab$

19. Given $x^2 - 2ax + b^2 = 0$

Transp. and compl. square, $x^2 - 2ax + a^2 = a^2 - b^2$

Evolving, $x - a = \pm\sqrt{a^2 - b^2}$

Whence, $x = a \pm \sqrt{a^2 - b^2}$

20. Given $\frac{2x(a-x)}{3a-2x} = \frac{a}{4}$

Clearing of fractions, $8ax - 8x^2 = 3a^2 - 2ax$

Whence, $8x^2 - 10ax = -3a^2$

Multiplying by 8, $64x^2 - 80ax = -24a^2$

Compl. the square, $64x^2 - 80ax + 25a^2 = a^2$

Evolving, $8x - 5a = \pm a$

Whence, $8x = 6a$, or $4a$

Dividing by 8, $x = \frac{3a}{4}$, or $\frac{a}{2}$

21. Given $6x + \frac{35 - 3x}{x} - 44 = 0$

Clearing of fract., $6x^2 + 35 - 3x - 44x = 0$

Or, $6x^2 - 47x = -35$

Multiplying by 24, $144x^2 - 1128x = -840$

Compl. the square, $144x^2 - 1128x + 47^2 = 1369$

Evolving, $12x - 47 = \pm 37$

Whence, $12x = 84$, or 10

Dividing by 12, $x = 7$, or $\frac{5}{6}$

22. Given $3x = 10 + \frac{x^2}{4}$

Clearing of fractions, $12x = 40 + x^2$

Or, $x^2 - 12x = -40$

Compl. the square, $x^2 - 12x + 36 = -4$

Evolving, $x - 6 = \pm 2\sqrt{-1}$

Whence, $x = 6 \pm 2\sqrt{-1}$

23. Given $\frac{3}{4}(x^2-3)=\frac{x-3}{8}$

Clearing of fractions, etc., $6x^2-18=x-3$

Or, $6x^2-x=15$

Dividing by 6, $x^2-\frac{1}{6}x=\frac{15}{6}$

Completing the square, $x^2-\frac{1}{6}x+\frac{1}{144}=\frac{361}{144}$

Evolving, $x-\frac{1}{12}=\pm\frac{19}{12}$

Whence, $x=\frac{5}{3}$, or $-\frac{3}{2}$

24. Given $x^2-4ax+a^2+b^2=0$

Compl. the square, $x^2-4ax+4a^2=3a^2-b^2$

Evolving, $x-2a=\pm\sqrt{3a^2-b^2}$

Whence, $x=2a\pm\sqrt{3a^2-b^2}$

25. Given $3x-\frac{x^2}{4}+3=14$

Clearing of fractions, etc., $12x-x^2=44$

Completing the square, $x^2-12x+36=-8$

Evolving, $x-6=\pm\sqrt{-8}$

Whence, $x=6\pm\sqrt{-8}$

26. Given $x^2+bx+cx=a^2-ab+ac-bc$

Completing the square,

$$x^2+(b+c)x+\left(\frac{b+c}{2}\right)^2=\frac{4a^2-4ab+4ac-2bc+b^2+c^2}{4}$$

Evolving, $x+\frac{b+c}{2}=\pm\frac{2a-b+c}{2}$

Whence, $x=\frac{-b-c\pm(2a-b+c)}{2}$

Or, $x=a-b$, or $-a-c$

27. Given $abx^2+\frac{3a^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}-\frac{b^2x}{c}$

Clearing of fract., $abc^2x^2+3a^2cx=6a^2+ab-2b^2-b^2cx$

Or, $abc^2x^2+(3a^2c+b^2c)x=6a^2+ab-2b^2$

Multiplying by $4ab$,

$$4a^2b^2c^2x^2+4abc(3a^2+b^2)x=24a^3b+4a^2b^2-8ab^3$$

Completing the square,

$$4a^2b^2c^2x^2+4abc(3a^2+b^2)x+(3a^2+b^2)^2$$
$$=9a^4+24a^3b+10a^2b^2-8ab^3+b^4$$

Evolving, $2abcx+3a^2+b^2=\pm(3a^2+4ab-b^2)$

Whence, $2abcx=-3a^2-b^2\pm(3a^2+4ab-b^2)$

Or, $2abcx=4ab-2b^2$, or $-(6a^2+4ab)$

Dividing by $2abc$, $x=\frac{2a-b}{ac}$, or $-\frac{3a+2b}{bc}$

28. Given $x^2-8x=14$

Completing the square, $x^2-8x+16=30$

Evolving, $x-4=\pm\sqrt{30}=\pm 5.4772$

Whence, $x=9.4772$, or -1.4772

29. Given $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$

Dividing by $7-4\sqrt{3}$, $x^2+\frac{2-\sqrt{3}}{7-4\sqrt{3}}x=\frac{2}{7-4\sqrt{3}}$

Or (Arts. 250, 251), $x^2+(2+\sqrt{3})x=2(7+4\sqrt{3})$

Completing square,

$$x^2+(2+\sqrt{3})x+\left(\frac{2+\sqrt{3}}{2}\right)^2=\frac{9(7+4\sqrt{3})}{4}$$

Evolving, $x+\frac{2+\sqrt{3}}{2}=\pm\frac{3}{2}(7+4\sqrt{3})^{\frac{1}{2}}$

Or (Art. 265), $x+\frac{1}{2}(2+\sqrt{3})=\pm\frac{3}{2}(2+\sqrt{3})$

Whence, $x=2+\sqrt{3}$, or $-2(2+\sqrt{3})$

30. Given $\frac{x+a}{x-2a}+\frac{x-2a}{x+a}=1$

Clear. of fract., $x^2+2ax+a^2+x^2-4ax+4a^2=x^2-ax-2a^2$

Or, $x^2-ax=-7a^2$

Compl. the square, $x^2 - ax + \frac{a^2}{4} = -\frac{27a^2}{4}$

Evolving, $x - \frac{a}{2} = \pm \frac{a}{2}\sqrt{-27}$

Whence, $x = \frac{a}{2}(1 \pm \sqrt{-27})$

PROBLEMS

LEADING TO AFFECTED QUADRATIC EQUATIONS.

(Pp. 217-221.)

2. Let $x =$ one number,

and $15 - x =$ the other.

Then, $x^2 + (15 - x)^2 = 117$

Or, $x^2 + 225 - 30x + x^2 = 117$

Whence, $2x^2 - 30x = -108$

Or, $x^2 - 15x = -54$

Compl. the square, $x^2 - 15x + \frac{225}{4} = \frac{9}{4}$

Evolving, $x - \frac{15}{2} = \pm \frac{3}{2}$

Whence, $x = 9$, or 6

Then $15 - x = 6$, or 9

3. Let $x =$ no. cords in one,

and $26 - x =$ no. cords in other.

Then, $x^2 + (26 - x)^2 = 356$

Or, $x^2 + 676 - 52x + x^2 = 356$

Whence, $2x^2 - 52x = -320$

Or, $x^2 - 26x = -160$

Compl. the square, $x^2 - 26x + 169 = 9$

Evolving, $x - 13 = \pm 3$

Whence, $x = 16$, or 10

Then, $26 - x = 10$, or 16

5. Let $x =$ the number bought,

and $\frac{240}{x} =$ cost per head.

Then, $(x-3)\left(\frac{240}{x}+8\right) = 240 + 59$

Or, $(x-3)(240+8x) = 299x$

Developing, $8x^2 + 216x - 720 = 299x$

Or, $8x^2 - 83x = 720$

Dividing by 8, $x^2 - \frac{83}{8}x = \frac{720}{8}$

Compl. the square, $x^2 - \frac{83}{8}x + (\frac{83}{16})^2 = \frac{29929}{256}$

Evolving, $x - \frac{83}{16} = \pm \frac{173}{16}$

Whence, $x = 16$, or $-\frac{45}{8}$

6. Let $x =$ width of frame,

Then $18 + 2x =$ length of glass incl. frame,

and $12 + 2x =$ width of glass incl. frame.

Therefore, $(18+2x)(12+2x) = 2 \times 18 \times 12$

Developing, $216 + 60x + 4x^2 = 432$

Or, $4x^2 + 60x = 216$

Dividing by 4, $x^2 + 15x = \frac{216}{4}$

Compl. square, $x^2 + 15x + \frac{225}{4} = \frac{441}{4}$

Evolving, $x + \frac{15}{2} = \pm \frac{21}{2}$

Whence, $x = 3$, or -18

7. Let $x =$ cost of the flour,

and $\frac{x^2}{100} =$ the gain.

Then, $x + \frac{x^2}{100} = 39$

Clearing of fract., $x^2 + 100x = 3900$

Compl. the square, $x^2 + 100x + 2500 = 6400$

Evolving, $x + 50 = \pm 80$

Whence, $x = 30$, or -130

8. Let $x =$ the larger number,

and $x - 9 =$ the smaller number.

Then, $(2x - 9)x = 266$

Or, $2x^2 - 9x = 266$

Multiplying by 8, $16x^2 - 72x = 2128$

Compl. the square, $16x^2 - 72x + 81 = 2209$

Evolving, $4x - 9 = \pm 47$

Whence, $4x = 56$, or -38

Dividing by 4, $x = 14$, or $-9\frac{1}{2}$

Then, $x - 9 = 5$, or $-18\frac{1}{2}$

9. Let $x =$ A's principal,

and $2600 - x =$ A's gain.

A's gain will be to B's gain as $12x$ to 16×3000, or as x to 4000;

hence $\frac{10400000 - 4000x}{x} =$ B's gain.

Therefore, $2600 - x + \frac{10400000 - 4000x}{x} = 1800$

Or, $2600x - x^2 + 10400000 - 4000x = 1800x$

Reducing, $x^2 + 3200x = 10400000$

Compl. square, $x^2 + 3200x + (1600)^2 = 12960000$

Evolving, $x + 1600 = \pm 3600$

Whence, $x = 2000$, or -5200

10. Let $x =$ no. barrels bought,

and $\frac{72}{x} =$ price paid per barrel.

By the conditions, $\frac{72}{x+6} = \frac{72}{x} - 1$

Clearing of fractions, $72x = 72x + 432 - x^2 - 6x$

Or, $x^2 + 6x = 432$

Compl. the square, $x^2 + 6x + 9 = 441$

Evolving, $x + 3 = \pm 21$

Whence, $x = 18, \text{ or } -24$

Also, $\frac{72}{x} = 4, \text{ or } -3$

11. Let $x =$ side of the court,

and $x^2 =$ area of the court.

Then $\frac{x+2}{6} =$ breadth of walk,

$\frac{4x+2}{3} =$ side of court, incl. walk,

and $4x =$ perimeter of court.

Therefore, $\left(\frac{4x+2}{3}\right)^2 - x^2 = 4x + 164$

Reducing, $\frac{7x^2+16x+4}{9} = 4x + 164$

Clearing of fract., $7x^2 + 16x + 4 = 36x + 1476$

Or, $7x^2 - 20x = 1472$

Multiplying by 7, $49x^2 - 140x = 10304$

Compl. square, $49x^2 - 140x + 100 = 10404$

Evolving, $7x - 10 = \pm 102$

Whence, $7x = 112, \text{ or } -92$

Dividing by 7, $x = 16, \text{ or } -13\frac{1}{7}$

Squaring, $x^2 = 256$

The above solution will involve smaller numbers if x is used to represent the breadth of the walk, and $6x - 2$ the side of the court. This will produce the equation $7x^2 - 8x = 39$, from which $x = 3$.

12. Let $\frac{x}{100} =$ rate of income tax.

and $10x =$ am't of income tax.

Then $(1000 - 10x)\left(\frac{x-1}{100}\right) =$ am't of second deduction.

Therefore, $1000 - 10x - \frac{(100-x)(x-1)}{10} = 912$

Clearing of fractions,

$$10000 - 100x - 101x + 100 + x^2 = 9120$$

Or, $x^2 - 201x = -980$

Completing square, $x^2 - 201x + \frac{40401}{4} = \frac{36481}{4}$

Evolving, $x - \frac{201}{2} = \pm \frac{191}{2}$

Whence, $x = 196$, or 5

It is evident that 196 per cent. is inadmissible, as that would require me to pay nearly twice my whole income as an income tax.

The above equation may be made to take a simple form by remembering that after the first deduction there is left $(100 - x)$ per cent. of \$1000, and after the second $(101 - x)$ per cent. of the previous remainder; hence, $\frac{(100-x)(101-x)}{10} = 912$, which reduces to the same form as before.

13. Let $x =$ no. hills in breadth,

and $x + 75 =$ no. hills in length.

Then, $x^2 + 75x = 6250$

Compl. square, $x^2 + 75x + (\frac{75}{2})^2 = \frac{30625}{4}$

Evolving, $x + \frac{75}{2} = \pm \frac{175}{2}$

Whence, $x = 50$, or -125

Therefore, $x + 75 = 125$, or -50

14. Let $x =$ no. persons at first,

and $x - 2 =$ no. persons afterwards.

Therefore, $\frac{6300}{x-2} = \frac{6300}{x} + 200$

Or, $\frac{63}{x-2} = \frac{63}{x} + 2$

Clearing of fractions, $63x = 63x - 126 + 2x^2 - 4x$

Or, $2x^2 - 4x = 126$
Dividing by 2, $x^2 - 2x = 63$
Completing square, $x^2 - 2x + 1 = 64$
Evolving, $x - 1 = \pm 8$
Whence, $x = 9$, or -7

15. Let $x =$ no. in front at first,
and $x + 6 =$ no. in depth at first.
Then $x^2 + 6x =$ whole no. men,
and $x + 870 =$ no. in front at last.
Therefore, $x^2 + 6x = 4(x + 870)$
Or, $x^2 + 2x = 3480$
Compl. square, $x^2 + 2x + 1 = 3481$
Evolving, $x + 1 = \pm 59$
Whence, $x = 58$, or -60
Therefore, $x^2 + 6x = 3712$

16. Let $x =$ side of smaller,
and $x + 4 =$ side of larger.
Then, $x^2 + (x + 4)^2 = 208$
Or, $2x^2 + 8x + 16 = 208$
Transp. and div. by 2, $x^2 + 4x = 96$
Completing square, $x^2 + 4x + 4 = 100$
Evolving, $x + 2 = \pm 10$
Whence, $x = 8$, or -12
Then, $x + 4 = 12$, or -8
Therefore, $(x + 4)^2 - x^2 = 144 - 64 = 80$

17. Let $x =$ breadth, in rods,
and $2x =$ length, in rods.
Then $2x^2 =$ area, in rods.
Therefore, $(2x + 20)(x + 24) = 4x^2$

Developing, $2x^2 + 68x + 480 = 4x^2$
Or, $2x^2 - 68x = 480$
Dividing by 2, $x^2 - 34x = 240$
Compl. square, $x^2 - 34x + 289 = 529$
Evolving, $x - 17 = \pm 23$
Whence, $x = 40, \text{ or } -6$
Therefore, $2x^2 = 3200$
3200 sq. rods = 20 acres.

18. Let $x =$ width of walk.
Then $6x - 1 =$ side of the court,
$(6x - 1)^2 =$ area of the court,
and $24x - 4 =$ perimeter of court.
Also $(24x - 4)x + 4x^2 =$ area of walk.

By the conditions, $28x^2 - 4x = 24x - 4 + 340$
Or, $28x^2 - 28x = 336$
Dividing by 28, $x^2 - x = 12$
Compl. square, $x^2 - x + \frac{1}{4} = \frac{49}{4}$
Evolving, $x - \frac{1}{2} = \pm \frac{7}{2}$
Whence, $x = 4, \text{ or } -3$
Also, $(6x - 1)^2 = 529$

This problem may also be solved in the same manner as the 11th.

19. Let $2x =$ no. bushels barley,
and $x - 4 =$ its price per bushel.

Then, $54x + 2x(x - 4) = 10(54 + 2x) + 576$
Or, $54x + 2x^2 - 8x = 540 + 20x + 576$
Transp. and uniting, $2x^2 + 26x = 1116$
Dividing by 2, $x^2 + 13x = 558$
Compl. square, $x^2 + 13x + \frac{169}{4} = \frac{2401}{4}$
Evolving, $x + \frac{13}{2} = \pm \frac{49}{2}$

Whence, $x = 18, \text{ or } -31$
Therefore, $2x = 36$
Also, $x - 4 = 14$
$14 \text{ dimes} = \$1.40$

20. Let $x =$ side of the smaller,
and $x + 1 =$ side of the larger.
Then, $x^2 + (x+1)^2 = 85$
Or, $2x^2 + 2x + 1 = 85$
Whence, $x^2 + x = 42$
Compl. square, $x^2 + x + \frac{1}{4} = \frac{169}{4}$
Evolving, $x + \frac{1}{2} = \pm \frac{13}{2}$
Whence, $x = 6, \text{ or } -7$
Also, $x + 1 = 7, \text{ or } -6$

$6^2 \div \frac{40}{36} = \frac{162}{5} =$ no. yards of carpeting required for smaller room; $\frac{162}{5} \times \$1.75 = \$56.70$.

$7^2 \div \frac{40}{36} = \frac{441}{10} =$ no. yards of carpeting required for larger room; $\frac{441}{10} \times \$1.75 = \$77.17\frac{1}{2}$.

EQUATIONS IN THE QUADRATIC FORM.

(ARTS. **282–284**, pp. 221–228.)

3. Given $x^4 + 4x^2 = 117$
Completing the square, $x^4 + 4x^2 + 4 = 121$
Extracting square root, $x^2 + 2 = \pm 11$
Whence, $x^2 = 9, \text{ or } -13$
Extracting square root, $x = \pm 3, \text{ or } \pm \sqrt{-13}$

4. Given $x^{-4} - 9x^{-2} + 20 = 0$
Completing square, $x^{-4} - 9x^{-2} + \frac{81}{4} = \frac{1}{4}$
Extracting square root, $x^{-2} - \frac{9}{2} = \pm \frac{1}{2}$

Whence, $x^{-2} = 5, \text{ or } 4$

Extracting square root, $x^{-1} = \pm\sqrt{5}, \text{ or } \pm 2$

Therefore, $x = \pm\frac{1}{\sqrt{5}}, \text{ or } \pm\frac{1}{2}$

Or, $x = \pm\frac{1}{5}\sqrt{5}, \text{ or } \pm\frac{1}{2}$

5. Given $x^{10} + 31x^5 - 10 = 22$

Multiplying by 4, etc., $4x^{10} + 124x^5 = 128$

Completing square, $4x^{10} + 124x^5 + 961 = 1089$

Extracting square root, $2x^5 + 31 = \pm 33$

Whence, $2x^5 = 2, \text{ or } -64$

Dividing by 2, $x^5 = 1, \text{ or } -32$

Extracting fifth root, $x = 1, \text{ or } -2$

6. Given $81x^2 + x^{-2} = 82$

Adding 18, $81x^2 + 18 + x^{-2} = 100$

Extracting sq. root, $9x + x^{-1} = \pm 10$

Multiplying by x, $9x^2 + 1 = \pm 10x$

Or, $9x^2 \mp 10x = -1$

Adding $\left(\frac{10}{2\sqrt{9}}\right)^2$, $9x^2 \mp 10x + \frac{25}{9} = \frac{16}{9}$

Extracting square root, $3x \mp \frac{5}{3} = \pm\frac{4}{3}$

Whence, $3x = \pm\frac{5}{3} \pm \frac{4}{3} = \pm 3, \text{ or } \pm\frac{1}{3}$

Dividing by 3, $x = \pm 1, \text{ or } \pm\frac{1}{3}$

The above example may also be solved like the 3d, after multiplying the given equation by x^2.

7. Given $x^2 + \frac{1225}{x^2} - 14 = 60$

Or, $x^2 + \frac{1225}{x^2} = 74$

Adding $2(x^2)^{\frac{1}{2}}(1225x^{-2})^{\frac{1}{2}}$, $x^2 + 70 + \frac{1225}{x^2} = 144$

Extracting square root, $x + \frac{35}{x} = \pm 12$

Multiplying by x, $x^2 + 35 = \pm 12x$

Or, $x^2 \mp 12x = -35$

Completing square, $x^2 \mp 12x + 36 = 1$

Extracting square root, $x \mp 6 = \pm 1$

Whence, $x = \pm 6 \pm 1$

Or, $x = \pm 7$, or ± 5

If the given equation be multiplied by x^2, it reduces to $x^4 - 74x^2 = -1225$, which is readily solved like the 3d.

8. Given $x^6 + 20x^3 - 10 = 59$

Completing square, $x^6 + 20x^3 + 100 = 169$

Extracting square root, $x^3 + 10 = \pm 13$

Whence, $x^3 = 3$, or -23

Extracting cube root, $x = \sqrt[3]{3}$, or $\sqrt[3]{-23}$

9. Given $3x^{2n} - 2x^n = 25$

Multiplying by 3, $9x^{2n} - 6x^n = 75$

Completing square, $9x^{2n} - 6x^n + 1 = 76$

Extracting square root, $3x^n - 1 = \pm\sqrt{76}$

Whence, $3x^n = 1 \pm 2\sqrt{19}$

Dividing by 3, $x^n = \frac{1 \pm 2\sqrt{19}}{3}$

Extracting nth root, $x = \sqrt[n]{\frac{1 \pm 2\sqrt{19}}{3}}$

12. Given $3x^{\frac{4}{3}} - \frac{5x^{\frac{8}{3}}}{2} = -592$

Clearing of fractions, etc., $5x^{\frac{8}{3}} - 6x^{\frac{4}{3}} = 1184$

Multiplying by 5, $25x^{\frac{8}{3}} - 30x^{\frac{4}{3}} = 5920$

Completing square, $25x^{\frac{8}{3}} - 30x^{\frac{4}{3}} + 9 = 5929$

Extracting square root, $5x^{\frac{4}{3}} - 3 = \pm 77$

Whence, $5x^{\frac{4}{3}} = 80, \text{ or } -74$

Dividing by 5, $x^{\frac{4}{3}} = 16, \text{ or } -\frac{74}{5}$

Raising to $\frac{3}{4}$th power, $x = 16^{\frac{3}{4}}, \text{ or } (-\frac{74}{5})^{\frac{3}{4}}$

Or, $x = 8, \text{ or } \sqrt[4]{(-\frac{74}{5})^3}$

13. Given $x^3 - x^{\frac{3}{2}} = 56$

Completing square, $x^3 - x^{\frac{3}{2}} + \frac{1}{4} = \frac{225}{4}$

Extracting square root, $x^{\frac{3}{2}} - \frac{1}{2} = \pm \frac{15}{2}$

Whence, $x^{\frac{3}{2}} = 8, \text{ or } -7$

Raising to $\frac{2}{3}$d power, $x = 8^{\frac{2}{3}}, \text{ or } (-7)^{\frac{2}{3}}$

Or, $x = 4, \text{ or } \sqrt[3]{49}$

14. Given $x^n - 2ax^{\frac{n}{2}} = b$

Completing square, $x^n - 2ax^{\frac{n}{2}} + a^2 = a^2 + b$

Extracting square root, $x^{\frac{n}{2}} - a = \pm \sqrt{a^2 + b}$

Whence, $x^{\frac{n}{2}} = a \pm \sqrt{a^2 + b}$

Raising to $\frac{2}{n}$th power, $x = (a \pm \sqrt{a^2 + b})^{\frac{2}{n}}$

15. Given $\sqrt{x^3} - 2\sqrt{x} - x = 0\sqrt{x}$

Dividing by $\sqrt{x}$, $x - 2 - \sqrt{x} = 0$

Or, $x - x^{\frac{1}{2}} = 2$

Completing square, $x - x^{\frac{1}{2}} + \frac{1}{4} = \frac{9}{4}$

Extracting square root, $x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2}$

Whence, $x^{\frac{1}{2}} = 2, \text{ or } -1$

Squaring, $x = 4, \text{ or } 1$

16. Given $$\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$$

Clearing of fractions, $$2x+2\sqrt{x}=16-x$$

Or, $$3x+2\sqrt{x}=16$$

Multiplying by 3, $$9x+6x^{\frac{1}{2}}=48$$

Completing square, $$9x+6x^{\frac{1}{2}}+1=49$$

Extracting square root, $$3x^{\frac{1}{2}}+1=\pm 7$$

Whence, $$3x^{\frac{1}{2}}=6, \text{ or } -8$$

Dividing by 3, $$x^{\frac{1}{2}}=2, \text{ or } -\tfrac{8}{3}$$

Squaring, $$x=4, \text{ or } \tfrac{64}{9}$$

17. Given $$x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$$

Completing square, $$x^{\frac{6}{5}}+x^{\frac{3}{5}}+\tfrac{1}{4}=\tfrac{3025}{4}$$

Extracting square root, $$x^{\frac{3}{5}}+\tfrac{1}{2}=\pm\tfrac{55}{2}$$

Whence, $$x^{\frac{3}{5}}=27, \text{ or } -28$$

Raising to $\frac{5}{3}$d power, $$x=27^{\frac{5}{3}}, \text{ or } (-28)^{\frac{5}{3}}$$

Or, $$x=243, \text{ or } -28^{\frac{5}{3}}$$

18. Given $$\frac{\frac{3\sqrt{x}}{5}-2}{x-5}=\frac{1}{20}$$

Multiplying by 5, $$\frac{3\sqrt{x}-10}{x-5}=\frac{1}{4}$$

Clearing of fractions, $$12\sqrt{x}-40=x-5$$

Or, $$x-12\sqrt{x}=-35$$

Completing square, $$x-12\sqrt{x}+36=1$$

Extracting square root, $$\sqrt{x}-6=\pm 1$$

Whence, $\sqrt{x} = 7, \text{ or } 5$
Squaring, $x = 49, \text{ or } 25$

22. Given $x + 16 - 7\sqrt{x+16} = 10 - 4\sqrt{x+16}$
Or, $x + 16 - 3\sqrt{x+16} = 10$
Compl. square, $(x+16) - 3(x+16)^{\frac{1}{2}} + \frac{9}{4} = \frac{49}{4}$
Extracting sq. root, $(x+16)^{\frac{1}{2}} - \frac{3}{2} = \pm \frac{7}{2}$
Whence, $(x+16)^{\frac{1}{2}} = 5, \text{ or } -2$
Squaring, $x + 16 = 25, \text{ or } 4$
Whence, $x = 9, \text{ or } -12$

The above equation may also be solved like the 7th, Art. 280.

23. Given $9x + \sqrt{16x^2 + 36x^3} = 15x^2 - 4$
Or, $4 + 9x + 2x(4 + 9x)^{\frac{1}{2}} = 15x^2$
Compl. square, $(4+9x) + 2x(4+9x)^{\frac{1}{2}} + x^2 = 16x^2$
Extracting sq. root, $(4+9x)^{\frac{1}{2}} + x = \pm 4x$
Whence, $(4+9x)^{\frac{1}{2}} = 3x, \text{ or } -5x$
Squaring, $4 + 9x = 9x^2, \text{ or } 25x^2$

From the positive root.

Transposing, etc., $9x^2 - 9x = 4$
Multiplying by 4, $36x^2 - 36x = 16$
Adding $\left(\frac{36}{2\sqrt{36}}\right)^2$, $36x^2 - 36x + 9 = 25$
Evolving, $6x - 3 = \pm 5$
Whence, $6x = 8, \text{ or } -2$
Dividing by 6, $x = \frac{4}{3}, \text{ or } -\frac{1}{3}$

From the negative root.

Transposing, etc., $25x^2 - 9x = 4$

Adding $\left(\frac{9}{2\sqrt{25}}\right)^2$, $25x^2 - 9x + \frac{81}{100} = \frac{481}{100}$

Evolving, $5x - \frac{9}{10} = \pm\frac{\sqrt{481}}{10}$

Whence, $5x = \frac{9 \pm \sqrt{481}}{10}$

Dividing by 5, $x = \frac{9 \pm \sqrt{481}}{50}$

24. Given $x^2(x^2-4)^{-2} + \frac{6}{x^2-4} = \frac{351x^{-2}}{25}$

Multiplying by $(x^2-4)^2$, $x^2 + 6(x^2-4) = \frac{351(x^2-4)^2}{25x^2}$

Or, $7x^2 - 24 = \frac{351(x^2-4)^2}{25x^2}$

Clearing of fractions, etc.,

$$175x^4 - 600x^2 = 351x^4 - 2808x^2 + 5616$$

Or, $176x^4 - 2208x^2 = -5616$

Dividing by 16, $11x^4 - 138x^2 = -351$

Dividing by 11, $x^4 - \frac{138}{11}x^2 = -\frac{351}{11}$

Compl. square, $x^4 - \frac{138}{11}x^2 + \frac{4761}{121} = \frac{900}{121}$

Extracting sq. root, $x^2 - \frac{69}{11} = \pm\frac{30}{11}$

Whence, $x^2 = 9$, or $\frac{39}{11}$

Evolving, $x = \pm 3$, or $\pm\sqrt{\frac{39}{11}}$

Or, $x = \pm 3$, or $\pm\frac{1}{11}\sqrt{429}$

25. Given $x^4 + 2ax^3 + 5a^2x^2 + 4a^3x = d$

Or, $(x^2 + ax)^2 + 4a^2(x^2 + ax) = d$

Compl. square, $(x^2 + ax)^2 + 4a^2(x^2 + ax) + 4a^4 = 4a^4 + d$

Extracting sq. root, $x^2+ax+2a^2=\pm\sqrt{4a^4+d}$

Transp. and mult. by 4, $4x^2+4ax=-8a^2\pm4\sqrt{4a^4+d}$

Compl. square, $4x^2+4ax+a^2=-7a^2\pm4\sqrt{4a^4+d}$

Evolving, $2x+a=\pm\sqrt{-7a^2\pm4\sqrt{4a^4+d}}$

Whence, $2x=-a\pm\sqrt{-7a^2\pm4\sqrt{4a^4+d}}$

Or, $x=-\frac{a}{2}\pm\frac{1}{2}\sqrt{-7a^2\pm4\sqrt{4a^4+d}}$

26. Given $x^3-8x^2+19x-12=0$

Multiplying by x, $x^4-8x^3+19x^2-12x=0$

Or, $(x^2-4x)^2+3(x^2-4x)=0$

Compl. square, $(x^2-4x)^2+3(x^2-4x)+\frac{9}{4}=\frac{9}{4}$

Extract. square root, $x^2-4x+\frac{3}{2}=\pm\frac{3}{2}$

Whence, $x^2-4x=0$, or -3

From the positive root.

Transposing, $x^2=4x$

Dividing by x, $x=4$

From the negative root.

Completing the square, $x^2-4x+4=1$

Evolving, $x-2=\pm1$

Whence, $x=3$, or 1

27. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$

Adding 9, $(2x^2+3x+9)-5(2x^2+3x+9)^{\frac{1}{2}}=6$

Compl. sq., $(2x^2+3x+9)-5(2x^2+3x+9)^{\frac{1}{2}}+\frac{25}{4}=\frac{49}{4}$

Ext. sq. root, $(2x^2+3x+9)^{\frac{1}{2}}-\frac{5}{2}=\pm\frac{7}{2}$

Whence, $(2x^2+3x+9)^{\frac{1}{2}}=6$, or -1

Squaring, $2x^2+3x+9=36$, or 1

Or, $2x^2+3x=27, \text{ or } -8$

Mult. by 8, $16x^2+24x=216, \text{ or } -64$

Compl. square, $16x^2+24x+9=225, \text{ or } -55$

Evolving, $4x+3=\pm 15, \text{ or } \pm\sqrt{-55}$

Whence, $4x=12, \text{ or } -18, \text{ or } -3\pm\sqrt{-55}$

Dividing by 4, $x=3, \text{ or } -\frac{9}{2}, \text{ or } \frac{-3\pm\sqrt{-55}}{4}$

28. Given $\sqrt[m]{(1+x)^2}-\sqrt[m]{1-x^2}=\sqrt[m]{(1-x)^2}$

Dividing by $\sqrt[m]{(1-x)^2}$, $\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}=1$

Compl. square, $\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}+\frac{1}{4}=\frac{5}{4}$

Extract. square root, $\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}-\frac{1}{2}=\pm\frac{1}{2}\sqrt{5}$

Whence, $\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}=\frac{1\pm\sqrt{5}}{2}$

Raising to *m*th power, $\frac{1+x}{1-x}=\left(\frac{1\pm\sqrt{5}}{2}\right)^m$

Clearing of fract., $2^m+2^m x=(1\pm\sqrt{5})^m-x(1\pm\sqrt{5})^m$

Transp., etc., $[(1\pm\sqrt{5})^m+2^m]x=(1\pm\sqrt{5}^m-2^m$

Whence, $x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}$

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

CASE I.

(ART. **287**, pp. 229, 230.)

2. Given $\begin{cases} 2x^2+xy-5y^2=20 & (1) \\ 2x-3y=1 & (2) \end{cases}$

From (2), $x=\frac{3y+1}{2} \qquad (3)$

Substituting (3) in (1), $\frac{9y^2+6y+1}{2}+\frac{3y^2+y}{2}-5y^2=20$

Clearing of fractions, $9y^2+6y+1+3y^2+y-10y^2=40$

Uniting terms, $2y^2+7y=39$

Dividing by 2, $y^2+\frac{7}{2}y=\frac{39}{2}$

Completing the square, $y^2+\frac{7}{2}y+\frac{49}{16}=\frac{361}{16}$

Evolving, $y+\frac{7}{4}=\pm\frac{19}{4}$

Whence, $y=3$, or $-6\frac{1}{2}$

Substituting in (3), $x=5$, or $-9\frac{1}{4}$

3. Given
$$\begin{cases} x+\frac{1}{3}=\frac{2x+y}{3} & (1) \\ \frac{x+y}{x}=\frac{4x-y}{2} & (2) \end{cases}$$

Clearing (1) of fractions, $3x+1=2x+y$ (3)

Or, $y=x+1$ (4)

Clearing (2) of fractions, $2x+2y=4x^2-xy$ (5)

Or, $4x^2-xy-2x=2y$ (6)

Substituting (4) in (6), $4x^2-x^2-x-2x=2x+2$

Or, $3x^2-5x=2$

Dividing by 3, $x^2-\frac{5}{3}x=\frac{2}{3}$

Completing the square, $x^2-\frac{5}{3}x+\frac{25}{36}=\frac{49}{36}$

Evolving, $x-\frac{5}{6}=\pm\frac{7}{6}$

Whence, $x=2$, or $-\frac{1}{3}$

Substituting in (4), $y=3$, or $\frac{2}{3}$

CASE II.

(ART. 288, pp. 230, 231.)

2. Given
$$\begin{cases} 3y^2-x^2=39 & (1) \\ x^2+4xy=256-4y^2 & (2) \end{cases}$$

Let $y=vx$ (3)

Substituting in (1), $3v^2x^2-x^2=39$ (4)

Substituting in (2), $x^2+4vx^2=256-4v^2x^2 \quad (5)$

From (4), $x^2=\dfrac{39}{3v^2-1} \quad (6)$

From (5), $x^2=\dfrac{256}{4v^2+4v+1} \quad (7)$

Hence, $\dfrac{39}{3v^2-1}=\dfrac{256}{4v^2+4v+1}$

Clearing of fract., $156v^2+156v+39=768v^2-256$

Or, $612v^2-156v=295$

Dividing by 612, $v^2-\frac{13}{51}v=\frac{295}{612}$

Compl. square, $v^2-\frac{13}{51}v+(\frac{13}{102})^2=\frac{5184}{10404}$

Evolving, $v-\frac{13}{102}=\pm\frac{72}{102}$

Whence, $v=\frac{5}{6}$, or $-\frac{59}{102}$

Substituting in (6), $x^2=36$, or 10404

Evolving, $x=\pm 6$, or ± 102

Substituting in (3), $y=\frac{5}{6}(\pm 6)$, or $-\frac{59}{102}(\pm 102)$

Or, $y=\pm 5$, or ∓ 59

3. Given $\begin{cases} 2x^2+3xy=26 & (1) \\ 3y^2+2xy=39 & (2) \end{cases}$

FIRST SOLUTION.

Dividing (1) by (2), $\dfrac{x(2x+3y)}{y(2x+3y)}=\dfrac{26}{39} \quad (3)$

Or, $\dfrac{x}{y}=\dfrac{2}{3} \quad (4)$

Whence, $y=\dfrac{3x}{2} \quad (5)$

Substituting in (1), $2x^2+\dfrac{9x^2}{2}=26$

Whence, $13x^2=52$

Dividing by 13, $x^2=4$

Evolving, $x=\pm 2$

Substituting in (5), $y=\pm 3$

SECOND SOLUTION.

Let $y = vx$ (3)

Substituting in (1), $2x^2 + 3vx^2 = 26$ (4)

Substituting in (2), $3v^2x^2 + 2vx^2 = 39$ (5)

From (4), $x^2 = \frac{26}{2+3v}$ (6)

From (5), $x^2 = \frac{39}{3v^2+2v}$ (7)

Hence, $\frac{26}{2+3v} = \frac{39}{3v^2+2v}$

Or, $\frac{2}{3v+2} = \frac{3}{3v^2+2v}$

Clearing of fractions, $6v^2 + 4v = 9v + 6$

Or, $6v^2 - 5v = 6$

Div. by 6 and compl. sq., $v^2 - \frac{5}{6}v + \frac{25}{144} = \frac{169}{144}$

Evolving, $v - \frac{5}{12} = \pm \frac{13}{12}$

Whence, $v = \frac{3}{2}$, or $-\frac{2}{3}$

Substituting in (6), $x^2 = 4$, or ∞

Evolving, $x = \pm 2$, or $\pm \infty$

Substituting in (3), $y = \pm 3$, or $\mp \infty$

The above equations may also be solved by finding the value of x, in terms of y, in (2) and substituting that value in (1). After developing the various expressions, clearing of fractions, and reducing, it will be found that y^4 disappears from the equation, and thus it becomes a pure quadratic, in which $y^2 = 9$.

4. Given $\begin{cases} x^2 + xy + 4y^2 = 6 & (1) \\ 3x^2 + 8y^2 = 14 & (2) \end{cases}$

Let $y = vx$ (3)

Substituting in (1), $x^2 + vx^2 + 4v^2x^2 = 6$ (4)

Substituting in (2), $3x^2 + 8v^2x^2 = 14$ (5)

From (4), $x^2 = \frac{6}{1+v+4v^2}$ (6)

From (5), $x^2=\frac{14}{3+8v^2}$ (7)

Hence, $\frac{14}{3+8v^2}=\frac{6}{1+v+4v^2}$

Or, $\frac{7}{3+8v^2}=\frac{3}{1+v+4v^2}$

Clearing of fract., $7+7v+28v^2=9+24v^2$

Or, $4v^2+7v=2$

Dividing by 4, $v^2+\frac{7}{4}v=\frac{2}{4}$

Compl. the square, $v^2+\frac{7}{4}v+\frac{49}{64}=\frac{81}{64}$

Evolving, $v+\frac{7}{8}=\pm\frac{9}{8}$

Whence, $v=\frac{1}{4}$, or -2

Substituting in (7), $x^2=4$, or $\frac{2}{5}$

Evolving, $x=\pm 2$, or $\pm\frac{1}{5}\sqrt{10}$

Substituting in (3), $y=\pm\frac{1}{2}$, or $\mp\frac{2}{5}\sqrt{10}$

CASE III.

(ARTS. **289, 290,** pp. 231–235.)

5. Given $\begin{cases} x^3+y^3=35 & (1) \\ x+y=5 & (2) \end{cases}$

Dividing (1) by (2), $x^2-xy+y^2=7$ (3)

Squaring (2), $x^2+2xy+y^2=25$ (4)

Subtracting (3) from (4), $3xy=18$ (5)

Whence, $xy=6$ (6)

Subtracting (6) from (3), $x^2-2xy+y^2=1$ (7)

Extr. square root of (7), $x-y=\pm 1$ (8)

Equation (2), $x+y=5$

Adding (2) and (8), $2x=6$, or 4

Subtracting (8) from (2), $2y=4$, or 6

Whence, $x=3$, or 2

Also, $y=2$, or 3

6. Given $\left\{\begin{array}{ll} x^2+y^2=20 & (1) \\ x^2-y^2=12 & (2) \end{array}\right.$

Adding (1) and (2), $2x^2=32$

Whence, $x^2=16$

Evolving, $x=\pm 4$

Subtracting (2) from (1), $2y^2=8$

Whence, $y^2=4$

Evolving, $y=\pm 2$

7. Given $\left\{\begin{array}{ll} x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y=19 & (1) \\ x^2+xy+y^2=133 & (2) \end{array}\right.$

Dividing (2) by (1), $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y=7$ (3)

Adding (1) and (3), $2x+2y=26$ (4)

Or, $x+y=13$ (5)

Squaring (5), $x^2+2xy+y^2=169$ (6)

Subtracting (2) from (6), $xy=36$ (7)

Multiplying by 4, $4xy=144$ (8)

Subtracting (8) from (6), $x^2-2xy+y^2=25$ (9)

Extract. sq. root of (9), $x-y=\pm 5$ (10)

Adding (5) and (10), $2x=18$, or 8

Subtracting (10) from (5), $2y=8$, or 18

Whence, $x=9$, or 4

Also, $y=4$, or 9

8. Given $\left\{\begin{array}{ll} x+y=9 & (1) \\ \sqrt[3]{x}+\sqrt[3]{y}=3 & (2) \end{array}\right.$

Subs. v for $\sqrt[3]{x}$, and z for $\sqrt[3]{y}$, $\left\{\begin{array}{ll} v^3+z^3=9 & (3) \\ v+z=3 & (4) \end{array}\right.$

Dividing (3) by (4), $v^2-vz+z^2=3$ (5)

Squaring (4), $v^2+2vz+z^2=9$ (6)

Subtracting (5) from (6), $3vz=6$ (7)

Whence, $vz = 2$ (8)

Subtracting (8) from (5), $v^2 - 2vz + z^2 = 1$ (9)

Extracting sq. root of (9), $v - z = \pm 1$ (10)

Adding (4) and (10), $2v = 4, \text{ or } 2$

Whence, $x^{\frac{1}{3}} = v = 2, \text{ or } 1$

Cubing, $x = 8, \text{ or } 1$

Subtracting (10) from (4), $2z = 2, \text{ or } 4$

Whence, $y^{\frac{1}{3}} = z = 1, \text{ or } 2$

Cubing, $y = 1, \text{ or } 8$

9. Given $\begin{cases} x^2 + y^2 + x + y = 18 & (1) \\ xy = 6 & (2) \end{cases}$

Multiplying (2) by 2, $2xy = 12$ (3)

Adding (1) and (3), $(x+y)^2 + (x+y) = 30$ (4)

Compl. square, $(x+y)^2 + (x+y) + \frac{1}{4} = \frac{121}{4}$ (5)

Extr. sq. root, $x + y + \frac{1}{2} = \pm \frac{11}{2}$ (6)

Whence, $x + y = 5, \text{ or } -6$ (7)

Squaring (7), $x^2 + 2xy + y^2 = 25, \text{ or } 36$ (8)

Subtract. (2) × 4 from (8), $x^2 - 2xy + y^2 = 1, \text{ or } 12$ (9)

Extracting sq. root of (9), $x - y = \pm 1, \text{ or } \pm 2\sqrt{3}$ (10)

Adding (7) and (10), $2x = 6, \text{ or } 4, \text{ or } -6 \pm 2\sqrt{3}$

Subtracting (10) from (7), $2y = 4, \text{ or } 6, \text{ or } -6 \mp 2\sqrt{3}$

Whence, $x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3}$

Also, $y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}$

10. Given $\begin{cases} x^2 + y^2 - x - y = 18 & (1) \\ xy + x + y = 19 & (2) \end{cases}$

Mult. (2) by 2, $2xy + 2(x+y) = 38$ (3)

Adding (1) and (3), $(x+y)^2 + (x+y) = 56$ (4)

Completing the square, $(x+y)^2 + (x+y) + \frac{1}{4} = \frac{225}{4}$ (5)

Extracting square root, $x+y+\frac{1}{2}=\pm\frac{15}{2}$ (6)

Whence, $x+y=7, \text{ or } -8$ (7)

Subtracting (7) from (2), $xy=12, \text{ or } 27$ (8)

Squaring (7), $x^2+2xy+y^2=49, \text{ or } 64$ (9)

Subtr. (8) × 4 from (9), $x^2-2xy+y^2=1, \text{ or } -44$ (10)

Extr. square root of (10), $x-y=\pm 1, \text{ or } \pm 2\sqrt{-11}$ (11)

Adding (7) and (11), $2x=8, \text{ or } 6, \text{ or } -8\pm 2\sqrt{-11}$

Subtracting (11) from (7), $2y=6, \text{ or } 8, \text{ or } -8\mp 2\sqrt{-11}$

Whence, $x=4, \text{ or } 3, \text{ or } -4\pm\sqrt{-11}$

Also, $y=3, \text{ or } 4, \text{ or } -4\mp\sqrt{-11}$

11. Given $$\begin{cases} x^2+3x+y=73-2xy & (1) \\ y^2+3y+x=44 & (2) \end{cases}$$

Adding (1) and (2), $x^2+y^2+4x+4y=117-2xy$ (3)

Or, $(x+y)^2+4(x+y)=117$ (4)

Compl. square, $(x+y)^2+4(x+y)+4=121$ (5)

Extr. square root, $x+y+2=\pm 11$ (6)

Whence, $x+y=9, \text{ or } -13$ (7)

From (7), $x=9-y, \text{ or } -13-y$ (8)

Substituting (8) in (2), $y^2+2y=35, \text{ or } 57$

Completing square, $y^2+2y+1=36, \text{ or } 58$

Evolving, $y+1=\pm 6, \text{ or } \pm\sqrt{58}$

Whence, $y=5, \text{ or } -7, \text{ or } -1\pm\sqrt{58}$

Substituting in (8), $x=4, \text{ or } 16, \text{ or } -12\mp\sqrt{58}$

PROMISCUOUS EXAMPLES.

(ART. 291, pp. 236, 237.)

1. Given $$\begin{cases} x-y=3 & (1) \\ x^2+y^2=117 & (2) \end{cases}$$

Squaring (1), $x^2-2xy+y^2=9$ (3)
Subtracting (3) from (2), $2xy=108$ (4)
Adding (4) and (2), $x^2+2xy+y^2=225$ (5)
Extr. square root of (5), $x+y=\pm 15$ (6)
Adding (1) and (6), $2x=18$, or -12
Subtracting (1) from (6) $2y=12$, or -18
Whence, $x=9$, or -6
Also, $y=6$, or -9

2. Given $\begin{cases} 2(x-y)=11 & (1) \\ xy=20 & (2) \end{cases}$

Dividing (1) by 2, $x-y=\frac{11}{2}$ (3)
Squaring (3), $x^2-2xy+y^2=\frac{121}{4}$ (4)
Adding (2) × 4 and (4), $x^2+2xy+y^2=\frac{441}{4}$ (5)
Extract. square root of (5), $x+y=\pm\frac{21}{2}$ (6)
Adding (3) and (6), $2x=16$, or -5
Subtracting (3) from (6), $2y=5$, or -16
Whence, $x=8$, or $-2\frac{1}{2}$
Also, $y=2\frac{1}{2}$, or -8

3. Given $\begin{cases} x-y=2 & (1) \\ x^3-y^3=98 & (2) \end{cases}$

Dividing (2) by (1), $x^2+xy+y^2=49$ (3)
Squaring (1), $x^2-2xy+y^2=4$ (4)
Subtracting (4) from (3), $3xy=45$ (5)
Or, $xy=15$ (6)
Adding (6) and (3), $x^2+2xy+y^2=64$ (7)
Extr. sq. root of (7), $x+y=\pm 8$ (8)
Adding (1) and (8), $2x=10$, or -6
Subtracting (1) from (8), $2y=6$, or -10
Whence, $x=5$, or -3
Also, $y=3$, or -5

4. Given $\begin{cases} 10x + y = 3xy & (1) \\ y - x = 2 & (2) \end{cases}$

From (2), $y = x + 2$ (3)

Subst. (3) in (1), $10x + x + 2 = 3x^2 + 6x$

Or, $3x^2 - 5x = 2$

Dividing by 3, $x^2 - \frac{5}{3}x = \frac{2}{3}$

Compl. square, $x^2 - \frac{5}{3}x + \frac{25}{36} = \frac{49}{36}$

Evolving, $x - \frac{5}{6} = \pm\frac{7}{6}$

Whence, $x = 2$, or $-\frac{1}{3}$

Substituting in (3), $y = 4$, or $\frac{5}{3}$

5. Given $\begin{cases} x^2 + xy = 84 & (1) \\ xy + y^2 = 60 & (2) \end{cases}$

Adding (1) and (2), $x^2 + 2xy + y^2 = 144$ (3)

Extracting sq. root of (3), $x + y = \pm 12$ (4)

Dividing (1) by (4), $x = \pm 7$

Dividing (2) by (4), $y = \pm 5$

6. Given $\begin{cases} \sqrt{x^2 + y^2} = 10 & (1) \\ x^2 - y^2 = 28 & (2) \end{cases}$

Squaring (1), $x^2 + y^2 = 100$ (3)

Adding (2) and (3), $2x^2 = 128$ (4)

Subtracting (2) from (3), $2y^2 = 72$ (5)

Dividing (4) by 2, $x^2 = 64$

Dividing (5) by 2, $y^2 = 36$

Whence, $x = \pm 8$

Also, $y = \pm 6$

7. Given $\begin{cases} x^2y + xy^2 = 30 & (1) \\ x^4y^2 + x^2y^4 = 468 & (2) \end{cases}$

Squaring (1), $x^4y^2 + 2x^3y^3 + x^2y^4 = 900$ (3)

Subtr. (2) from (3), $2x^3y^3 = 432$ (4)

Whence, $x^3y^3 = 216$ (5)
Extr. cube root of (5), $xy = 6$ (6)
Dividing (1) by (6), $x + y = 5$ (7)
Squaring (7), $x^2 + 2xy + y^2 = 25$ (8)
Subtracting (6) × 4 from (8), $x^2 - 2xy + y^2 = 1$ (9)
Extr. sq. root of (9), $x - y = \pm 1$ (10)
Adding (7) and (10), $2x = 6$, or 4
Subtracting (10) from (7), $2y = 4$, or 6
Whence, $x = 3$, or 2
Also, $y = 2$, or 3

8. Given $\begin{cases} x^2 + y^2 = 25 & (1) \\ 2xy = 24 & (2) \end{cases}$

Adding (1) and (2), $x^2 + 2xy + y^2 = 49$ (3)
Subtracting (2) from (1), $x^2 - 2xy + y^2 = 1$ (4)
Extr. sq. root of (3), $x + y = \pm 7$ (5)
Extr. sq. root of (4), $x - y = \pm 1$ (6)
Adding (5) and (6), $2x = \pm 8$, or ± 6
Subtracting (6) from (5), $2y = \pm 6$, or ± 8
Whence, $x = \pm 4$, or ± 3
Also, $y = \pm 3$, or ± 4

9. Given $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2} & (1) \\ x^2y + xy^2 = 162 & (2) \end{cases}$

From (1), $x + y = \frac{1}{2}xy$ (3)
Dividing (2) by (3), $xy = \dfrac{324}{xy}$ (4)
Clearing (4) of fractions, $x^2y^2 = 324$ (5)
Extr. square root, $xy = \pm 18$ (6)
Substituting (6) in (3), $x + y = \pm 9$ (7)
Squaring (7), $x^2 + 2xy + y^2 = 81$ (8)
Subtracting (6) × 4 from (8), $x^2 - 2xy + y^2 = 9$, or 153 (9)

Extr. square root of (9), $x-y=\pm 3$, or $\pm 3\sqrt{17}$ (10)

Equation (7), $x+y=9$, or -9

Adding (7) and (10), $2x=12$, or 6, or $-9\pm 3\sqrt{17}$

Subtracting (10) from (7), $2y=6$, or 12, or $-9\mp 3\sqrt{17}$

Whence, $x=6$, or 3, or $\frac{3}{2}(-3\pm\sqrt{17})$

Also, $y=3$, or 6, or $\frac{3}{2}(-3\mp\sqrt{17})$

10. Given
$$\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3} & (1) \\ x^2+y^2=45 & (2) \end{cases}$$

Clearing (1) of fract., $3(x+y)^2+3(x-y)^2=10(x^2-y^2)$ (3)

Or, $6x^2+6y^2=10x^2-10y^2$ (4)

Transp. and uniting terms in (4), $16y^2=4x^2$ (5)

Whence, $-x^2+4y^2=0$ (6)

Adding (2) and (6), $5y^2=45$

Dividing by 5, $y^2=9$

Evolving, $y=\pm 3$

Substituting in (6), $x^2=36$

Evolving, $x=\pm 6$

11. Given
$$\begin{cases} x^2+y^2=\frac{5}{2}xy & (1) \\ x-y=\frac{1}{4}xy & (2) \end{cases}$$

Squaring (2), $x^2-2xy+y^2=\dfrac{x^2y^2}{16}$ (3)

From (1), $x^2-\frac{5}{2}xy+y^2=0$ (4)

Subtracting (4) from (3), $\frac{1}{2}xy=\dfrac{x^2y^2}{16}$ (5)

Or, $8xy=x^2y^2$ (6)

Dividing by xy, $xy=8$ (7)

Substituting (7) in (2), $x-y=2$ (8)

Squaring (8), $x^2-2xy+y^2=4$ (9)

Adding (7) × 4 and (9), $x^2+2xy+y^2=36$ (10)

Extr. sq. root of (10), $x+y=\pm 6$ (11)

Adding (8) and (10), $2x=8, \text{ or } -4$

Subtracting (8) from (10), $2y=4, \text{ or } -8$

Whence, $x=4, \text{ or } -2$

Also, $y=2, \text{ or } -4$

12. Given $\begin{cases} x+y=a & (1) \\ x^2+y^2=b & (2) \end{cases}$

Squaring (1), $x^2+2xy+y^2=a^2$ (3)

Subtracting (2) from (3), $2xy=a^2-b$ (4)

Subtracting (4) from (2), $x^2-2xy+y^2=2b-a^2$ (5)

Extract. sq. root of (5), $x-y=\pm\sqrt{2b-a^2}$ (6)

Adding (1) and (6), $2x=a\pm\sqrt{2b-a^2}$

Subtracting (6) from (1), $2y=a\mp\sqrt{2b-a^2}$

Whence, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2b-a^2}$

Also, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{2b-a^2}$

13. Given $\begin{cases} 3x^2-2xy=15 & (1) \\ 2x+3y=12 & (2) \end{cases}$

From (2), $y=\dfrac{12-2x}{3}$ (3)

Substituting (3) in (1), $3x^2-\dfrac{24x-4x^2}{3}=15$ (4)

Whence, $13x^2-24x=45$

Multiplying by 13, $169x^2-312x=585$

Completing square, $169x^2-312x+144=729$

Evolving, $13x-12=\pm 27$

Whence, $13x=39, \text{ or } -15$

Dividing by 13, $x=3, \text{ or } -1\frac{2}{13}$

Substituting in (3), $y=2, \text{ or } 4\frac{10}{13}$

14. Given $$\begin{cases} 6x^2+2y^2=5xy+12 & (1)\\ 2xy+3x^2=3y^2-3 & (2)\end{cases}$$

Let $y=vx$ (3)

Substituting in (1), $6x^2+2v^2x^2=5vx^2+12$ (4)

Substituting in (2), $2vx^2+3x^2=3v^2x^2-3$ (5)

From (4), $x^2=\dfrac{12}{2v^2-5v+6}$ (6)

From (5), $x^2=\dfrac{3}{3v^2-2v-3}$ (7)

Hence, $\dfrac{4}{2v^2-5v+6}=\dfrac{1}{3v^2-2v-3}$

Clear. of fract., $12v^2-8v-12=2v^2-5v+6$

Whence, $10v^2-3v=18$

Dividing by 10, $v^2-\frac{3}{10}v=\frac{18}{10}$

Completing square, $v^2-\frac{3}{10}v+\frac{9}{400}=\frac{729}{400}$

Evolving, $v-\frac{3}{20}=\pm\frac{27}{20}$

Whence, $v=\frac{3}{2}$, or $-\frac{6}{5}$

Substituting in (6), $x^2=4$, or $\frac{25}{31}$

Evolving, $x=\pm 2$, or $\pm\frac{5}{31}\sqrt{31}$

Substituting in (3), $y=\pm 3$, or $\mp\frac{6}{31}\sqrt{31}$

15. Given $$\begin{cases} xz=y^2 & (1)\\ (x+y)(z-x-y)=3 & (2)\\ (x+y+z)(z-x-y)=7 & (3)\end{cases}$$

Dividing (2) by (3), $\dfrac{x+y}{x+y+z}=\dfrac{3}{7}$ (4)

Clearing of fractions, $7x+7y=3x+3y+3z$ (5)

Whence, $4x+4y=3z$ (6)

Or, $x+y=\dfrac{3z}{4}$ (7)

Substituting in (2), $\dfrac{3z}{4}\times\dfrac{z}{4}=3$ (8)

Whence, $3z^2=48$ (9)

Dividing by 3, $z^2 = 16$ (10)
Evolving, $z = \pm 4$ (11)
Substituting (11) in (1), $x = \dfrac{y^2}{\pm 4}$ (12)
Developing (3), $z^2 - (x + y)^2 = 7$ (13)
Substituting (11) in (13), $16 - (x + y)^2 = 7$ (14)
Or, $(x + y)^2 = 9$ (15)
Evolving, $x + y = \pm 3$ (16)
Substituting (12) in (16), $\dfrac{y^2}{\pm 4} + y = \pm 3$ (17)
Clearing of fractions, $y^2 \pm 4y = \pm 12$
Completing square, $y^2 \pm 4y + 4 = 16, \text{ or } -8$
Evolving, $y \pm 2 = \pm 4, \text{ etc.}$
Whence, $y = \pm 2, \text{ or } \pm 6$
Substituting in (16), $x = \pm 1, \text{ or } \mp 9$

When $y = 6$, then $x = -9$, and when $y = -6$, $x = 9$, that is, the values of x and y must have opposite signs, and it makes no difference whether we write $x = \pm 9$ and $y = \mp 6$, or $x = \mp 9$ and $y = \pm 6$.

PROBLEMS

LEADING TO QUADRATIC EQUATIONS.

(Pp. 237–240.)

1. Let $x = \text{A's money}$,
and $y = \text{B's money}$.
Then $x + y = 18$ (1)
and $2xy = 154$ (2)
Squaring (1), $x^2 + 2xy + y^2 = 324$ (3)
Subtracting (2) × 2 from (3), $x^2 - 2xy + y^2 = 16$ (4)
Extracting square root of (4), $x - y = \pm 4$ (5)
Adding (1) and (5), $2x = 22, \text{ or } 14$

Subtracting (5) from (1), $2y = 14$, or 22

Whence, $x = 11$, or 7

Also, $y = 7$, or 11

2. Let $x =$ the greater,

and $y =$ the less.

Then $x - y = 5 \qquad (1)$

and $x^2 + y^2 = 193 \qquad (2)$

Squaring (1), $x^2 - 2xy + y^2 = 25 \qquad (3)$

Multiplying (2) by 2, $2x^2 + 2y^2 = 386 \qquad (4)$

Subtracting (3) from (4), $x^2 + 2xy + y^2 = 361 \qquad (5)$

Extracting sq. root of (5), $x + y = \pm 19 \qquad (6)$

Adding (1) and (6), $2x = 24$, or -14

Subtracting (1) from (6), $2y = 14$, or -24

Whence, $x = 12$, or -7

Also, $y = 7$, or -12

3. Let $x =$ side of A's,

and $y =$ side of B's.

Then, $4x + 4y = 200 \qquad (1)$

and $x^2 + y^2 = 1300 \qquad (2)$

Dividing (1) by 4, $x + y = 50 \qquad (3)$

Squaring (3), $x^2 + 2xy + y^2 = 2500 \qquad (4)$

Multiplying (2) by 2, $2x^2 + 2y^2 = 2600 \qquad (5)$

Subtracting (4) from (5), $x^2 - 2xy + y^2 = 100 \qquad (6)$

Extracting sq. root of (6), $x - y = \pm 10 \qquad (7)$

Adding (3) and (7), $2x = 60$, or 40

Subtracting (7) from (3), $2y = 40$, or 60

Whence, $x = 30$, or 20

Also, $y = 20$, or 30

$$\$2\tfrac{1}{4} \times 20^2 = \$900\,; \quad \$2\tfrac{1}{4} \times 30^2 = \$2025$$

4. Let $x =$ the larger number,
and $y =$ the smaller number.
Then $xy = 77$ (1)
and $x^2 - y^2 : (x-y)^2 :: 9 : 2$ (2)
From (2), $2(x^2 - y^2) = 9(x-y)^2$ (3)
Dividing (3) by $x-y$, $2(x+y) = 9(x-y)$ (4)
From (4), $7x = 11y$ (5)
Or, $x = \frac{11y}{7}$ (6)
Substituting in (1), $\frac{11y^2}{7} = 77$
Dividing by $\frac{11}{7}$, $y^2 = 49$
Evolving, $y = \pm 7$
Substituting in (6), $x = \pm 11$

5. Let $x =$ A's age,
and $y =$ B's age.
Then $xy = 750$ (1)
and $(x-2)(y+7) = 851$ (2)
From (2), $xy + 7x - 2y = 865$ (3)
From (1), $x = \frac{750}{y}$ (4)
Substituting (4) in (3), $750 + \frac{5250}{y} - 2y = 865$ (5)
Or, $\frac{5250}{y} - 2y = 115$
Whence, $2y^2 + 115y = 5250$
Dividing by 2, $y^2 + \frac{115}{2}y = 2625$
Completing square, $y^2 + \frac{115}{2}y + (\frac{115}{4})^2 = \frac{55225}{16}$
Evolving, $y + \frac{115}{4} = \pm \frac{235}{4}$
Whence, $y = 30$, or $-87\frac{1}{2}$
Substituting in (4), $x = 25$, or $-8\frac{4}{7}$

6. Let $x =$ its length,

and $y =$ its breadth.

Then $2x + 2y =$ its perimeter,

and $\underline{14x + 14y + 7^2 \times 4} =$ area of walk.

Therefore, $xy = 15000$ (1)

Also, $\underline{14x + 14y + 196 = 3696}$ (2)

From (2), $x + y = 250$ (3)

Squaring (3), $x^2 + 2xy + y^2 = 62500$ (4)

Subtract. (1) × 4 from (4), $x^2 - 2xy + y^2 = 2500$ (5)

Extracting sq. root of (5), $x - y = \pm 50$ (6)

Adding (3) and (6), $2x = 300$, or 200

Subtracting (6) from (3), $2y = 200$, or 300

Whence, $x = 150$, or 100

Also, $y = 100$, or 150

7. Let $x =$ the greater,

and $y =$ the less.

Then, $(x - y)\, y = 42$ (1)

and $\underline{(x - y)(x + y) = 133}$ (2)

Dividing (2) by (1), $\dfrac{x + y}{y} = \dfrac{19}{6}$ (3)

Or, $6x = 13y$ (4)

Whence, $x = \dfrac{13y}{6}$ (5)

Substituting (5) in (1), $\dfrac{13y^2}{6} - y^2 = 42$

Whence, $7y^2 = 252$

Dividing by 7, $y^2 = 36$

Evolving, $y = \pm 6$

Substituting in (5), $x = \pm 13$

8. Let $x =$ price of a duck,

and $y =$ price of a turkey.

Then, $10x + 12y = 22\frac{1}{2}$ (1)

and $\frac{6}{x} - \frac{5}{y} = 4$ (2)

Clearing (2) of fractions, $6y - 5x = 4xy$ (3)

From (1), $x = \frac{45 - 24y}{20}$ (4)

Substituting (4) in (3), $6y - \frac{45 - 24y}{4} = \frac{45y - 24y^2}{5}$

Clearing of fractions, $120y - 225 + 120y = 180y - 96y^2$

Or, $96y^2 + 60y = 225$

Dividing by 96, $y^2 + \frac{5}{8}y = \frac{75}{32}$

Compl. the square, $y^2 + \frac{5}{8}y + \frac{25}{256} = \frac{625}{256}$

Evolving, $y + \frac{5}{16} = \pm \frac{25}{16}$

Whence, $y = \frac{5}{4}$, or $-\frac{15}{8}$

Substituting in (4), $x = \frac{3}{4}$, or $\frac{9}{2}$

$\$\frac{3}{4} = \$0.75$, and $\$\frac{5}{4} = \$1.25$.

9. Let $x =$ breadth of farm, in rods.

Then $4x =$ length of farm, in rods,

and $40x =$ price of farm.

Therefore, $\frac{4x^2}{160} \times x = 40x$

Or, $\frac{x^3}{40} = 40x$

Multiplying by $\frac{40}{x}$, $x^2 = 40^2$

Evolving, $x = 40$

Multiplying by 4 $4x = 160$

Multiplying by 40, $40x = 1600$

10. Let $x =$ length of one,

and $y =$ length of the other.

Then $6x^2 =$ surface of one,

and $6y^2 =$ surface of the other.

Therefore, $x+y=20$ (1)

and $x^3+y^3=2240$ (2)

Dividing (2) by (1), $x^2-xy+y^2=112$ (3)

Squaring (1), $x^2+2xy+y^2=400$ (4)

Subtracting (3) from (4), $3xy=288$ (5)

Whence, $xy=96$ (6)

Subtracting (6) from (3), $x^2-2xy+y^2=16$ (7)

Extracting sq. root of (7), $x-y=\pm 4$ (8)

Adding (1) and (8), $2x=24$, or 16

Subtracting (8) from (1), $2y=16$, or 24

Whence, $x=12$, or 8

Also, $y=8$, or 12

Therefore, $6x^2=864$, or 384

Also, $6y^2=384$, or 864

11. Let $x=$ A's share,

and $y=$ B's share,

Then $x+y=500$ (1)

and $297-x : 297-y :: 2x : 8y$ (2)

From (2), $2376y-8xy=594x-2xy$ (3)

Or, $2376y-594x=6xy$ (4)

Dividing by 6, $396y-99x=xy$ (5)

From (1), $x=500-y$ (6)

Substit. (6) in (5), $396y-49500+99y=500y-y^2$

Or, $y^2-5y=49500$

Compl. square, $y^2-5y+\frac{25}{4}=\frac{198025}{4}$

Evolving, $y-\frac{5}{2}=\pm\frac{445}{2}$

Whence, $y=225$, or -220

Substituting in (6), $x=275$, or 720

12. Let $x =$ number bought.

Then $\frac{300}{x} =$ price paid per head.

and $\frac{270}{x-15} =$ price rec'd per head.

Therefore, $\frac{300}{x} + \frac{1}{2} = \frac{270}{x-15}$

Clear. of fract., $600x - 9000 + x^2 - 15x = 540x$

Whence, $x^2 + 45x = 9000$

Compl. square, $x^2 + 45x + \frac{2025}{4} = \frac{38025}{4}$

Evolving, $x + \frac{45}{2} = \pm \frac{195}{2}$

Whence, $x = 75$, or -120

13. Let $x =$ first rate per cent.,

and $y =$ second rate per cent.

Then $\frac{3600}{y} =$ first sum,

and $\frac{4900}{x} =$ second sum.

By the conditions, $\frac{3600}{y} + \frac{4900}{x} = 1300$ (1)

Also, $\frac{36x}{y} = \frac{49y}{x}$ (2)

Clearing (1) of fract., etc., $36x + 49y = 13xy$ (3)

Clearing (2) of fractions, $36x^2 = 49y^2$ (4)

Extracting square root of (4), $6x = 7y$ (5)

Or, $x = \frac{7y}{6}$ (6)

Substituting (6) in (3), $42y + 49y = \frac{91y^2}{6}$

Or, $91y = \frac{91y^2}{6}$

Dividing by $\frac{91y}{6}$, etc., $y = 6$

Substituting in (6), $x = 7$

14. Let $x =$ A's number of acres,
and $y =$ B's number of acres.
Then $\frac{200}{x} =$ A's price per acre,
and $\frac{200}{y} =$ B's price per acre.

Therefore, $x + y = 200$ (1)
Also, $\frac{200}{x} - \frac{200}{y} = \frac{3}{4}$ (2)
Clearing (2) of fract., $800\,y - 800\,x = 3\,xy$ (3)
From (1), $y = 200 - x$ (4)
Substit. (4) in (3), $160000 - 800\,x - 800\,x = 600\,x - 3\,x^2$
Or, $3\,x^2 - 2200\,x = -160000$
Dividing by 3, $x^2 - \frac{2200}{3}x = -\frac{160000}{3}$
Compl. square, $x^2 - \frac{2200}{3}x + \left(\frac{1100}{3}\right)^2 = \frac{730000}{9}$
Evolving, $x - \frac{1100}{3} = \pm\frac{100}{3}\sqrt{73} = \pm\frac{854.4003}{3}$
Whence, $x = 651.467$, or 81.867
Substituting in (4) $y = -451.467$, or 118.133
Whence, $\frac{200}{x} = \frac{200}{81.867} = 2.443$
Also, $\frac{200}{y} = \frac{200}{118.133} = 1.693$

15. Let $x =$ the whole distance,
$y =$ A's rate per day,
and $z =$ B's rate per day.

Then $7\,y - 7\,z = y + 5$ (1)
And $10\,y + 10\,z = x + 100$ (2)
Also, $\frac{x}{y} + 3 = \frac{x}{z}$ (3)
Clearing (3) of fract., $xz + 3\,yz = xy$ (4)
From (1), $y = \frac{7\,z + 5}{6}$ (5)

Subst. (5) in (2), $\frac{35z+25}{3}+10z=x+100$ (6)

From (6), $65z-3x=275$ (7)

Subst. (5) in (4), $xz+\frac{7z^2+5z}{2}=\frac{7xz+5x}{6}$ (8)

From (8), $21z^2+15z=xz+5x$ (9)

From (7), $x=\frac{65z-275}{3}$ (10)

Subst. (10) in (9), $21z^2+15z=\frac{65z^2-275z}{3}+\frac{325z-1375}{3}$

Clear. of fract., etc., $63z^2+45z=65z^2+50z-1375$

Or, $2z^2+5z=1375$

Dividing by 2, $z^2+\frac{5}{2}z=\frac{1375}{2}$

Compl. square, $z^2+\frac{5}{2}z+\frac{25}{16}=\frac{11025}{16}$

Evolving, $z+\frac{5}{4}=\pm\frac{105}{4}$

Whence, $z=25$, or $-27\frac{1}{2}$

Substituting in (5), $y=30$

Substituting in (10), $x=450$

THEORY OF QUADRATIC EQUATIONS.

(ART. **295**, pp. 241, 242.)

3. Assume $x^2-6x+8=0$

Whence, $x^2-6x+9=1$

Evolving, $x-3=\pm 1$

Whence, $x=4$, or 2

Therefore, $x^2-6x+8=(x-4)(x-2)$

The factors may also be determined by inspection. See Art. 102.

4. Assume $x^2+73x+780=0$

Whence, $x^2+73x+(\frac{73}{2})^2=\frac{2209}{4}$

Evolving, $x+\frac{73}{2}=\pm\frac{47}{2}$

Whence, $x=-13$, or -60

Therefore, $x^2+73x+780=(x+60)(x+13)$

5. Assume $2x^2+x-6=0$

Or, $x^2+\frac{1}{2}x-3=0$

Whence, $x^2+\frac{1}{2}x+\frac{1}{16}=\frac{49}{16}$

Evolving, $x+\frac{1}{4}=\pm\frac{7}{4}$

Whence, $x=\frac{3}{2}$, or -2

Therefore, $2x^2+x-6=2(x+2)(x-\frac{3}{2})$

6. Assume $x^2+\frac{52}{3}x-\frac{256}{3}=0$

Whence, $x^2+\frac{52}{3}x+\frac{676}{9}=\frac{1444}{9}$

Evolving, $x+\frac{26}{3}=\pm\frac{38}{3}$

Whence, $x=4$, or $-\frac{64}{3}$

Therefore, $x^2+\frac{52}{3}x-\frac{256}{3}=(x-4)(x+\frac{64}{3})$

FORMATION OF EQUATIONS.

(ART. **296**, p. 243.)

2. $(x-1)(x+2)=x^2+x-2=0$

Whence, $x^2+x=2$

3. $(x-4)(x-5)=x^2-9x+20=0$

Whence, $x^2-9x=-20$

4. $[(x-(1+\sqrt{5})][x-(1-\sqrt{5})]=x^2-2x+(1-5)=0$

Whence, $x^2-2x=4$

5. $(x-3)(x+\frac{3}{5})=x^2-\frac{12}{5}x-\frac{9}{5}=0$

Whence, $x^2-\frac{12}{5}x=\frac{9}{5}$

Or, $5x^2-12x=9$

6. $(x-7)\left(x+\frac{19}{3}\right)=x^2-\frac{2}{3}x-\frac{133}{3}=0$

Whence, $x^2-\frac{2}{3}x=44\frac{1}{3}$

7. $[x-(m+\sqrt{n})][x-(m-\sqrt{n})]=x^2-2mx+(m^2-n)=0$

Whence, $x^2-2mx=n-m^2$

RATIO AND PROPORTION.

(PROBLEMS, pp. 257, 258.)

1. Given $x:18::6:27$

By Art. 311, $27x=108$

Whence, $x=4$

2. Given $\frac{1}{2}:\frac{1}{3}::\frac{1}{4}:x$

By Art. 311, $\frac{x}{2}=\frac{1}{12}$

Whence, $x=\frac{1}{6}$

3. Given $2:x::x:8$

By Art. 311, $x^2=16$

Whence, $x=4$

4. Let x and y represent the numbers.

Then $x:y::2\frac{1}{2}:2 \quad (1)$

Also, $x-5:y-5::1\frac{1}{3}:1 \quad (2)$

From (1), by Art. 320, $x:y::5:4 \quad (3)$

From (2), by Art. 320, $x-5:y-5::4:3 \quad (4)$

From (3), by Art. 311, $y=\frac{4x}{5} \quad (5)$

Substituting (5) in (4), $x-5:\frac{4x}{5}-5::4:3 \quad (6)$

From (6), by Art. 320, $5x - 25 : 4x - 25 :: 4 : 3$ (7)

By Art. 311, $15x - 75 = 16x - 100$

Whence, $x = 25$

Substituting in (5), $y = 20$

5. Let $x =$ the greater part,

and $y =$ the less.

Then $x + y = 50$ (1)

Also, $\underline{x + 3 : y - 3 :: 3 : 2}$ (2)

From (2), by Art. 315, $x + y : y - 3 :: 5 : 2$ (3)

Substituting (1) in (3), $50 : y - 3 :: 5 : 2$ (4)

From (4), by Art. 321, $10 : y - 3 :: 1 : 2$ (5)

By Art. 311, $y - 3 = 20$

Whence, $y = 23$

Substituting in (1), $x + 23 = 50$

Whence, $x = 27$

6. Let $x =$ am't of wine in 1 gal. of first,

and $y =$ am't of wine in 1 gal. of second.

Then $\frac{x}{1 - x} =$ ratio of wine to water in first,

and $\frac{y}{1 - y} =$ ratio of wine to water in second.

By the first condition, a mixture of 1 gallon of each will contain 1 gallon of wine and 1 of water; and by the second condition, a mixture of 4 gallons of the first and 1 gallon of the second will contain 2 gallons of wine and 3 gallons of water.

Therefore, $x + y = 1$ (1)

and $\underline{4x + y = 2}$ (2)

Subtracting (1) from (2), $3x = 1$

Whence, $x = \frac{1}{3}$

Substituting in (1), $\frac{1}{3} + y = 1$

Whence, $y=\frac{2}{3}$

Therefore, $\frac{x}{1-x}=\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{1}{2}=1:2$

Also, $\frac{y}{1-y}=\frac{\frac{2}{3}}{\frac{1}{3}}=\frac{2}{1}=2:1$

The second equation given above may also be obtained as follows:

$$4x+y:5-(4x+y)::2:3$$

By Art. 315, $4x+y:5::2:5$

By Art. 321, $4x+y:1::2:1$

By Art. 311, $4x+y=2$

The above example may also be solved like the 7th by using $\frac{1}{x}$ in place of x, and $\frac{1}{y}$ in place of y.

7. Let $\frac{1}{x}=$ proportion of males,

and $1-\frac{1}{x}=$ proportion of females.

Then $\frac{1}{x}:1-\frac{1}{x}=\frac{1}{x-1}=$ ratio of males to females.

Therefore, $\frac{98}{1000}\left(1-\frac{1}{x}\right)-\frac{46}{1000}\left(\frac{1}{x}\right)=\frac{18}{1000}$

Or, $98-\frac{98}{x}-\frac{46}{x}=18$

Whence, $\frac{144}{x}=80$

Dividing by 144, $\frac{1}{x}=\frac{80}{144}=\frac{5}{9}$

Therefore, $1-\frac{1}{x}=1-\frac{5}{9}=\frac{4}{9}$

Also, Art. 320, $\frac{1}{x}:1-\frac{1}{x}=\frac{1}{x-1}=\frac{5}{9}:\frac{4}{9}=5:4$

8. Let $x =$ rate of the first train,

and $y =$ rate of the second train.

Then, $-y : x+y :: 2 : 30$

By Art. 317, $2x : 2y :: 32 : 28$

By Art. 320, $x : y :: 8 : 7$

VARIATION.

(Art. 332, pp. 260, 261.)

1. Given $y = mx$

Subst. 2 for x and 10 for y, $10 = 2m$

Whence, $m = 5$

Therefore, $y = 5x$

2. Given $y = mx$

Subst. 2 for y and 1 for x, $2 = m$

Therefore, $y = 2x$

Substituting 2 for x, $y = 4$

3. Given $y = mz$

Subst. 24 for y and 3 for z, $24 = 3m$

Whence, $m = 8$

Therefore, $y = 8z$

4. Given $x = \frac{m}{y}$

Or, $m = xy$

Subst. 4 for x and 2 for y, $m = 8$

Therefore, $x = \frac{8}{y}$

Substituting 6 for y, $x = \frac{8}{6} = 1\frac{1}{3}$

5. Given $z = mxy$

Subst. 1 for x, y, and z, $m = 1$

Therefore, $z = xy$

Subst. 2 for x and y, $z = 2 \times 2 = 4$

6. Given $y = a + mxy \qquad (1)$

Subst. 2 for x and $-2\frac{1}{3}$ for y, $-\frac{7}{3} = a - \frac{14}{3}m \qquad (2)$

Subst. -2 for x and 1 for y, $1 = a - 2m \qquad (3)$

Subtracting (2) from (3), $\frac{10}{3} = \frac{8}{3}m$

Whence, $m = \frac{5}{4}$

From (3), $a = 1 + 2m$

Substituting value of m, $a = 1 + \frac{5}{2} = \frac{7}{2}$

Substituting in (1), $y = \frac{7}{2} + \frac{5}{4}xy$

Clearing of fractions, $4y = 14 + 5xy$

Or, $4y - 5xy = 14$

Whence, $y = \dfrac{14}{4 - 5x}$

7. Let $x =$ its diameter.

Then the areas are $36m$, $64m$, and mx^2.

Hence, $mx^2 = 36m + 64m$

Or, $x^2 = 36 + 64 = 100$

Evolving, $x = 10$

8. Let $x =$ the additional distance.

The two distances are then 3 and $3 + x$, and the amounts of light are, respectively, $\frac{m}{9}$ and $\frac{m}{(3+x)^2}$.

Hence, $\dfrac{m}{9} = \dfrac{2m}{(3+x)^2}$

Or, $(3+x)^2 = 18$

Evolving, $3 + x = \sqrt{18} = 3\sqrt{2}$

Whence, $x = 3\sqrt{2} - 3 = 3(\sqrt{2} - 1)$

Or, $x = 3 \times .4142 = 1.243$

9. Let $x =$ the number of cars which will just balance the power of the engine.

Then $x - 1 =$ the number it can move.

By the conditions, $\begin{cases} m\sqrt{x} = 24 & (1) \\ m\sqrt{4} = 4 & (2) \end{cases}$

From (2), $2\,m = 4$

Or, $m = 2$

Substituting in (1), $2\sqrt{x} = 24$

Dividing by 2, $\sqrt{x} = 12$

Squaring, $x = 144$

Then, $x - 1 = 143$

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

(Art. 338, pp. 264–266.)

It will be a valuable exercise for the student to obtain all the formulas found in the table, from the two *fundamental* ones.

2. $l = a + (n-1)\,d = 10 + 3 \times (-2) = 10 - 6 = 4.$

3. $$n = \frac{d - 2\,a \pm \sqrt{(d - 2\,a)^2 + 8\,d\,S}}{2\,d} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} + 1}}{-\frac{1}{6}}$$

$$= \frac{-\frac{3}{4} \pm \frac{5}{4}}{-\frac{1}{6}} = \frac{-9 \pm 15}{-2} = -3, \text{ or } 12.$$

4. $$d = \frac{2\,(S - a\,n)}{n\,(n-1)} = \frac{2\,(440 - 60)}{20 \times 19} = \frac{2 \times 380}{20 \times 19} = 2.$$

5. $$a = \frac{2\,S - n\,(n-1)\,d}{2\,n} = \frac{3\,n^2 + 4\,n - 3\,n\,(n-1)}{2\,n}$$

$$= \frac{3\,n + 4 - 3\,n + 3}{2} = \frac{7}{2}.$$

6. $d=\frac{l-a}{m+1}=\frac{18+9}{3+1}=\frac{27}{4}=6\frac{3}{4}$; $-9+6\frac{3}{4}=-2\frac{1}{4}$;
$-2\frac{1}{4}+6\frac{3}{4}=4\frac{1}{2}$; $4\frac{1}{2}+6\frac{3}{4}=11\frac{1}{4}$.

7. $a=\frac{2S-n(n-1)d}{2n}=\frac{642-6\times5\times5}{12}=\frac{642-150}{12}$
$=\frac{492}{12}=41.$

$l=\frac{642+150}{12}=\frac{792}{12}=66.$

8. $S=\frac{(l+a)(l-a+d)}{2d}=\frac{(99+1)(99-1+2)}{2\times2}$
$=\frac{100\times100}{4}=2500.$

Or, $S=\frac{1}{2}n(a+l)=25\times100=2500.$

9. $l=a+(n-1)d=1+51\times2=1+102=103.$
$S=\frac{1}{2}n(a+l)=26\times104=2704.$

10. $S=\frac{1}{2}n[2a+(n-1)d]$, and $S=180n$.

Therefore, $180n=\frac{n}{2}[540-60(n-1)]$

Dividing by $\frac{n}{2}$, etc., $360=540-60n+60$

Whence, $60n=240$

Dividing by 60, $n=4.$

11. $S=\frac{1}{2}n[2a+(n-1)d]$, and $S=10n$.

Therefore, $10n=\frac{n}{2}[16+\frac{1}{2}(n-1)]$

Dividing by $\frac{n}{2}$, etc., $20=16+\frac{n}{2}-\frac{1}{2}$

Clear. of fractions, $40=32+n-1$

Whence, $n=9$

Also, $S=10\times9=90.$

GEOMETRICAL PROGRESSION.

(ART. **344**, pp. 269–271.)

As in Arithmetical Progression, it will be well for the student to obtain all the formulas in the table from the two *fundamental* ones. Those involving logarithms will be obtained in Art. 420.

2. $S = \frac{ar^n - a}{r-1} = \frac{7 \times 3^5 - 7}{3-1} = \frac{1701 - 7}{2} = 847.$

3. $a = \frac{l}{r^{n-1}} = \frac{128}{2^5} = \frac{128}{32} = 4.$

4. $r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[7]{\frac{4374}{2}} = \sqrt[7]{2187} = 3.$

5. $r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \left(\frac{128}{\frac{1}{2}}\right)^{\frac{1}{4}} = (256)^{\frac{1}{4}} = 4.$

$\frac{1}{2} \times 4 = 2;\ 2 \times 4 = 8;\ 8 \times 4 = 32.$

6. $l = ar^{n-1} = 2 \times 4^{11} = 2 \times 4194304 = 8388608$

$S = \frac{lr - a}{r-1} = \frac{33554430}{3} = 11184810.$

7. $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1\frac{1}{2}.$

8. $S = \frac{ar^n - a}{r-1} = \frac{a - ar^n}{1-r} = \frac{\frac{3}{2} - \frac{3}{2} \times (\frac{2}{3})^6}{1-\frac{2}{3}} = \frac{\frac{3}{2} - \frac{32}{243}}{\frac{1}{3}}$

$= \frac{\frac{665}{486}}{\frac{1}{3}} = \frac{665}{162} = 4\frac{17}{162}.$

10. $S = \frac{a}{1-r} = \frac{.12}{1-.01} = \frac{12}{99} = \frac{4}{33}.$

11. $S = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$

12. $a = \frac{(r-1)S}{r^n - 1} = \frac{(\frac{3}{2} - 1)\,2059}{(\frac{3}{2})^7 - 1} = \frac{\frac{2059}{2}}{\frac{2059}{128}} = 64.$

13. $S = \frac{ar^n - a}{r - 1} = \frac{3 \times 3^9 - 3}{3 - 1} = \frac{59049 - 3}{2} = 29523.$

HARMONICAL PROGRESSION.

(Art. **347**, pp. 272, 273.)

1. $l = \frac{ab}{(n-1)a - (n-2)b} = \frac{\frac{1}{2}}{4 - \frac{3}{2}} = \frac{\frac{1}{2}}{\frac{5}{2}} = \frac{1}{5}.$

2. $d = \frac{a - l}{(m+1)al} = \frac{\frac{1}{6} - \frac{1}{12}}{3 \times \frac{1}{6} \times \frac{1}{12}} = \frac{\frac{1}{12}}{\frac{1}{24}} = 2.$ The reciprocals of the extremes are 6 and 12. $6 + 2 = 8$; $8 + 2 = 10$.

Hence, 6, 8, 10, 12 = the arithmetical progression,
and $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$ = the harmonical progression.

$d = \frac{a - l}{(m+1)al} = \frac{4 - 5}{3 \times 4 \times 5} = -\frac{1}{60}.$ The reciprocals of the extremes are $\frac{1}{4}$ and $\frac{1}{5}$; $\frac{1}{4} - \frac{1}{60} = \frac{7}{30}$; $\frac{7}{30} - \frac{1}{60} = \frac{13}{60}$.

Hence, $\frac{1}{4}$, $\frac{7}{30}$, $\frac{13}{60}$, $\frac{1}{5}$ = the arithmetical progression,
and 4, $4\frac{2}{7}$, $4\frac{8}{13}$, 5 = the harmonical progression.

3. The reciprocals of the terms, $\frac{1}{3}$, $\frac{5}{6}$, $\frac{4}{3}$, etc., form an arithmetical progression whose common difference is $\frac{1}{2}$. $\frac{4}{3} + \frac{1}{2} = \frac{11}{6}$; $\frac{11}{6} + \frac{1}{2} = \frac{14}{6}$. The required terms are the reciprocals of $\frac{11}{6}$ and $\frac{14}{6}$, that is, $\frac{6}{11}$ and $\frac{6}{14}$.

PROBLEMS

REQUIRING THE APPLICATION OF THE PRINCIPLES OF THE PROGRESSIONS.

(Pp. 273–276.)

3. Let x, $x + y$, $x + 2y$, and $x + 3y$ represent the numbers.

Then, $x^2 + (x + y)^2 = 34 \qquad (1)$

Also, $(x + 2y)^2 + (x + 3y)^2 = 130 \qquad (2)$

From (1), $2x^2+2xy+y^2=34$ (3)

From (2), $2x^2+10xy+13y^2=130$ (4)

Subtr. (3) from (4), $8xy+12y^2=96$ (5)

From (5), $x=\frac{24-3y^2}{2y}$ (6)

Subst. (6) in (3), $\frac{576-144y^2+9y^4}{2y^2}+24-3y^2+y^2=34$

Or, $\frac{576-144y^2+9y^4}{2y^2}=10+2y^2$

Clearing of fractions, $576-144y^2+9y^4=20y^2+4y^4$

Transposing and uniting, $5y^4-164y^2=-576$

Completing square, $y^4-\frac{164}{5}y^2+(\frac{82}{5})^2=\frac{3844}{25}$

Extracting square root, $y^2-\frac{82}{5}=\pm\frac{62}{5}$

Whence, $y^2=\frac{144}{5}$, or 4

Evolving, $y=\pm\frac{12}{5}\sqrt{5}$, or ± 2

Substituting in (6), $x=\mp\frac{13}{5}\sqrt{5}$, or ± 3

Therefore,
$$\begin{cases} x+y=\mp\frac{1}{5}\sqrt{5}, \text{ or } \pm 5 \\ x+2y=\pm\frac{11}{5}\sqrt{5}, \text{ or } \pm 7 \\ x+3y=\pm\frac{23}{5}\sqrt{5}, \text{ or } \pm 9 \end{cases}$$

4. Let x, $\frac{2xy}{x+y}$, and y represent the numbers.

Then $x+\frac{2xy}{x+y}+y=13$ (1)

Also, $xy=18$ (2)

Substituting (2) in (1), $x+y+\frac{36}{x+y}=13$ (3)

Clear. of fract., etc., $(x+y)^2-13(x+y)=-36$ (4)

Compl. square, $(x+y)^2-13(x+y)+\frac{169}{4}=\frac{25}{4}$ (5)

Evolving, $x+y-\frac{13}{2}=\pm\frac{5}{2}$ (6)

Whence, $x+y=9$, or 4 (7)

Squaring (7), $x^2+2xy+y^2=81$, or 16 (8)

Subtracting (2) × 4 from (8), $x^2-2xy+y^2=9, \text{ or } -56$ (9)
Extr. square root of (9), $x-y=\pm 3$ (10)
Adding (7) and (10), $2x=12, \text{ or } 6$
Subtracting (10) from (7), $2y=6, \text{ or } 12$
Whence, $x=6, \text{ or } 3$
Also, $y=3, \text{ or } 6$
Therefore, $\frac{2xy}{x+y}=\frac{36}{9}=4$

5. Here $a=25$, $d=-5$, and $n=11$. Therefore, $S=\frac{1}{2}n\,[2a+(n-1)\,d]=\frac{11}{2}\,[50-50]=0$. Hence he was back at Boston at the end of the 11th day.

6. Here $a=2$, $d=2$, and $n=6401$. Therefore $S=\frac{1}{2}n\,[2a+(n-1)\,d]=\frac{6401}{2}\,[4+12800]=\frac{6401}{2}\times 12804=40979202$. 40979202 rods = 128060 miles 2 rods.

7. The time of raising the first foot must be $\frac{1}{10}$ of 2 days, or $\frac{1}{5}$ day, and each foot will require $\frac{1}{5}$ day more than the preceding. Then $a=\frac{1}{5}$, $d=\frac{1}{5}$, and $n=100$. Therefore
$S=\frac{1}{2}n\,[2a+(n-1)\,d]=50\,[\frac{2}{5}+\frac{99}{5}]=50\times\frac{101}{5}=1010$.

8. Let $x-2y$, $x-y$, x, $x+y$, and $x+2y$, respectively, represent the number of days in which each, separately, can reap an acre.

Then, $5x=20$ (1)
Whence, $x=4$ (2)
Again, $\frac{1}{x-2y}+\frac{1}{x-y}+\frac{1}{x}+\frac{1}{x+y}+\frac{1}{x+2y}=\frac{87}{60}$ (3)
From (2) and (3), $\frac{1}{x-2y}+\frac{1}{x-y}+\frac{1}{x+y}+\frac{1}{x+2y}=\frac{72}{60}$ (4)
Uniting terms, $\frac{2x}{x^2-4y^2}+\frac{2x}{x^2-y^2}=\frac{72}{60}$ (5)
Dividing by $2x=8$, $\frac{1}{x^2-4y^2}+\frac{1}{x^2-y^2}=\frac{3}{20}$ (6)

Clear. of fract., etc., $20\,(2\,x^2-5\,y^2)=3\,(x^4-5\,x^2y^2+4\,y^4)$

Subst. value of x, $640-100\,y^2=768-240\,y^2+12\,y^4$

Or, $12\,y^4-140\,y^2=-128$

Dividing by 12, $y^4-\frac{70}{6}\,y^2=-\frac{64}{6}$

Completing square, $y^4-\frac{70}{6}\,y^2+(\frac{35}{6})^2=\frac{841}{36}$

Extr. square root, $y^2-\frac{35}{6}=\pm\frac{29}{6}$

Whence, $y^2=\frac{32}{3}$, or 1

Evolving, $y=\pm\frac{4}{3}\sqrt{6}$, or ± 1

Therefore,
$$\begin{cases} x-2\,y=4\mp\frac{8}{3}\sqrt{6}, \text{ or } 2, \text{ or } 6 \\ x-y=4\mp\frac{4}{3}\sqrt{6}, \text{ or } 3, \text{ or } 5 \\ x+y=4\pm\frac{4}{3}\sqrt{6}, \text{ or } 5, \text{ or } 3 \\ x+2\,y=4\pm\frac{8}{3}\sqrt{6}, \text{ or } 6, \text{ or } 2 \end{cases}$$

9. Here $a=100$, $r=0.9375=\frac{15}{16}$, and $n=\infty$. Substituting these values, we have $S=\frac{a}{1-r}=\frac{100}{1-\frac{15}{16}}=\frac{1600}{16-15}=$ 1600. Each of these terms, except the first, must be doubled to obtain the whole distance passed over; hence the distance is $1600\times 2-100=3100$.

10. Here $a=20$, $d=-5$, and $n=8$. Substituting these values we have $S=\frac{1}{2}\,n\,[2\,a+(n-1)\,d]=4\,(40-35)=4\times 5=20$.

11. Let $x-3\,y$, $x-y$, $x+y$, and $x+3\,y$ represent the numbers.

Then, $2\,x^2+18\,y^2=200$ (1)

Also, $2\,x^2+\ 2\,y^2=136$ (2)

Subtracting (2) from (1), $16\,y^2=\ 64$ (3)

Whence, $y^2=4$

Evolving, $y=\pm 2$

Substituting in (2) ÷ 2, $x^2+4=68$

Whence, $x^2 = 64$

Evolving, $x = \pm 8$

Therefore,
$$\left\{\begin{aligned} x - 3y &= \pm 2 \\ x - y &= \pm 6 \\ x + y &= \pm 10 \\ x + 3y &= \pm 14 \end{aligned}\right.$$

12. Let $x =$ the number of hours.

Then $4\left(x + \frac{11}{4}\right) = 4x + 11 =$ distance A travels,

and $\frac{x}{2}\left[9 + (x - 1)\frac{1}{4}\right] = \frac{35x}{8} + \frac{x^2}{8} =$ distance B travels.

Therefore, $4x + 11 = \frac{35x}{8} + \frac{x^2}{8}$

Clearing of fractions, $32x + 88 = 35x + x^2$

Whence, $x^2 + 3x = 88$

Completing square, $x^2 + 3x + \frac{9}{4} = \frac{361}{4}$

Evolving, $x + \frac{3}{2} = \pm \frac{19}{2}$

Whence, $x = 8$, or -11

13. Let x, xy, xy^2, and xy^3 represent the numbers.

Then, $x + xy = 15$ (1)

Also, $xy^2 + xy^3 = 60$ (2)

Dividing (2) by (1), $y^2 = 4$

Evolving, $y = \pm 2$

Substituting in (1), $x \pm 2x = 15$

Or, $3x$, or $-x = 15$

Whence, $x = 5$, or -15

Therefore,
$$\left\{\begin{aligned} xy &= 10, \text{ or } 30 \\ xy^2 &= 20, \text{ or } -60 \\ xy^3 &= 40, \text{ or } 120 \end{aligned}\right.$$

FIRST SOLUTION.

14. Let x, $\frac{2xy}{x+y}$, and y represent the numbers.

Then, $\frac{2x^2y}{x+y}=y$ (1)

Also, $x+\frac{2xy}{x+y}+2=y+1$ (2)

Clearing (1) of fract., $2x^2y=xy+y^2$ (3)

Dividing (3) by y, $2x^2=x+y$ (4)

Substituting (4) in (2), $x+\frac{2xy}{2x^2}+2=y+1$ (5)

Or, $x+\frac{y}{x}+1=y$ (6)

Clearing (6) of fract., $x^2+y+x=xy$ (7)

Or, $x^2=xy-(x+y)$ (8)

Adding (4) and (8), $3x^2=xy$ (9)

Whence, $3x=y$ (10)

Substituting (10) in (4), $2x^2=4x$

Whence, $x=2$

Substituting in (10), $y=6$

Therefore, $\frac{2xy}{x+y}=3$

SECOND SOLUTION.

Let x, y, and xy represent the numbers.

Then, Art. 345, $\frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{xy}$ (1)

Also, $x+y+2=xy+1$ (2)

From (1), $y=2x-1$ (3)

Substituting (3) in (2), $x+2x+1=2x^2-x+1$ (4)

Or, $2x^2=4x$

Whence, $x=2$

Substituting in (3), $y=4-1=3$

Therefore, $xy=6$

PERMUTATIONS AND COMBINATIONS.

(Art. **353**, pp. 279, 280.)

4. Here $n=8$, $r=4$, and $n-r+1=5$; then, by formula (1), $8\times 7\times 6\times 5=1680$.

5. Here $n=40$, $r=6$, and $n-r+1=35$; then, by formula (4),

$$\frac{40\times 39\times 38\times 37\times 36\times 35}{1\times 2\times 3\times 4\times 5\times 6}$$

$$=2\times 39\times 38\times 37\times 35=3838380$$

BINOMIAL THEOREM.

(Arts. **361–363**, pp. 285–288.)

4. Here $n=50$, and $r=49$; and substituting these in the general term (5 and 7, Art. 360), we have

$$\frac{50\,.\,49\,.\,48\,\ldots\,3}{1\,.\,2\,.\,3\,\ldots\,48}a^2x^{48}=\frac{50\,.\,49}{1\,.\,2}a^2x^{48}=1225\,a^2x^{48}$$

The coefficient may also be found by means of principle 6, Art. 360.

6. Here $n=10$, and $r=9$; and substituting these in the general term (7, Art. 360), we have

$$\frac{10\,.\,9\,.\,8\,\ldots\ldots\,3}{1\,.\,2\,.\,3\,\ldots\ldots\,8}x^2a^8=\frac{10\,.\,9}{1\,.\,2}x^2a^8=45\,x^2a^8$$

7. $(1+x)^6=1^6+6\,(1)^5x+15\,(1)^4x^2+20\,(1)^3x^3+15\,(1)^2x^4$
$+6\,(1)\,x^5+x^6$
$=1+6\,x+15\,x^2+20\,x^3+15\,x^4+6\,x^5+x^6$

9. Substituting 1 for a and x^2 for x in the 5th example, we obtain the required result.

10. Here $n = 8$, and $r = 5$; and substituting these in the general term (7, Art. 360), we have

$$\frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4}\,a^4 x^4 = 70\,m^8 n^{12}$$

11. $(3a^2 - 2b^3)^6 = (3a^2)^6 - 6\,(3a^2)^5\,(2b^3) + 15\,(3a^2)^4\,(2b^3)^2$
$- 20\,(3a^2)^3\,(2b^3)^3 + 15\,(3a^2)^2\,(2b^3)^4 - 6\,(3a^2)\,(2b^3)^5$
$+ (2b^3)^6 =$
$729\,a^{12} - 2916\,a^{10}b^3 + 4860\,a^8 b^6 - 4320\,a^6 b^9 + 2160\,a^4 b^{12}$
$- 576\,a^2 b^{15} + 64\,b^{18}$

12. Substituting 1 for a and $-\frac{1}{x}$ for x in the 5th example, we obtain the required result.

13. The exponent of b in the required term is 7; hence (2, Art. 360) the required term is the 8th. Substituting in the general term (7, Art. 360), we have

$$\frac{9\,.\,8\,.\,7\,.\,.\,.\,.\,.\,3}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,7}\,(a^3)^2\,(3\,a\,b)^7 = \frac{9\,.\,8}{1\,.\,2}\,a^6\,(2187\,a^7 b^7) = 78732\,a^{13} b^7$$

14. Here $n = 2002$, and $r = 2001$; and substituting these in the general term (7, Art. 360), we have

$$\frac{2002\,.\,2001\,.\,2000\,.\,.\,.\,.\,3}{1\,.\,2\,.\,3\,.\,.\,.\,.\,2000}\left(a^{\frac{3}{10}}\right)^2\left(x^{\frac{3}{10}}\right)^{2000} = \frac{2002 \times 2001}{1 \times 2}\,a^{\frac{3}{5}}\,x^{600}$$

15. $$\left(1 - \frac{x}{2}\right)^{10} = 1 - 10\left(\frac{x}{2}\right) + \frac{10 \times 9}{2}\left(\frac{x}{2}\right)^2 - \frac{45 \times 8}{3}\left(\frac{x}{2}\right)^3$$
$$+ \frac{120 \times 7}{4}\left(\frac{x}{2}\right)^4 - \frac{210 \times 6}{5}\left(\frac{x}{2}\right)^5 + \frac{252 \times 5}{6}\left(\frac{x}{2}\right)^6$$
$$- \frac{210 \times 4}{7}\left(\frac{x}{2}\right)^7 + \frac{120 \times 3}{8}\left(\frac{x}{2}\right)^8 - \frac{45 \times 2}{9}\left(\frac{x}{2}\right)^9 + \frac{10}{10}\left(\frac{x}{2}\right)^{10}$$
$$= 1 - 5x + \tfrac{45}{4}x^2 - 15x^3 + \tfrac{105}{8}x^4 - \tfrac{63}{8}x^5 + \tfrac{105}{32}x^6 - \tfrac{15}{16}x^7$$
$$+ \tfrac{45}{256}x^8 - \tfrac{5}{256}x^9 + \tfrac{1}{1024}x^{10}$$

17. It is evident that a term containing $a^2 b^2$ must be found in the 4th power of $a+b$; hence, considering the quantity as $(a+b)+c$, a term containing $a^2 b^2 c^2$ must be a part of that term which contains $(a+b)^4 c^2$, or the 3d term (2, Art. 360). Substituting in the general term,

$$\frac{6 \times 5}{1 \times 2}(a+b)^4 c^2 = 15\,(a+b)^4 c^2$$

By principles 2 and 7, Art. 360, the 3d term of $(a+b)^4$, or that which involves $a^2 b^2$, is $\frac{4 \times 3}{1 \times 2} a^2 b^2$, or $6\,a^2 b^2$. Multiplying this by $15\,c^2$ we obtain $90\,a^2 b^2 c^2$.

18. The formula for the third power is

$$(a - x)^3 = a^3 - 3\,a^2 x + 3\,a x^2 - x^3$$

Substituting $a+2\,b$ for a, and c for x, we have

$$\begin{aligned}(a+2\,b-c)^3 &= (a+2\,b)^3 - 3\,(a+2\,b)^2 c + 3\,(a+2\,b)\,c^2 - c^3 \\ &= a^3 + 6\,a^2 b + 12\,a b^2 + 8\,b^3 - 3\,a^2 c - 12\,a b c - 12\,b^2 c \\ &\quad + 3\,a c^2 + 6\,b c^2 - c^3\end{aligned}$$

SERIES.

UNDETERMINED COEFFICIENTS.

(Art. **369**, pp. 289–294.)

4. Assume $\frac{a}{b+x} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$

Clearing of fractions,

$$a = \begin{array}{l|l|l|l} Ab + Bb & x + Cb & x^2 + Db & x^3 + \ldots\ldots \\ \quad\;\; + A & \quad\;\; + B & \quad\;\; + C & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} Ab=a \\ A+Bb=0 \\ B+Cb=0 \\ C+Db=0 \end{array}\right\}\text{ whence }\left\{\begin{array}{l} A=\frac{a}{b} \\ B=-\frac{a}{b^2} \\ C=\frac{a}{b^3} \\ D=-\frac{a}{b^4} \end{array}\right.$$

Substituting, we have

$$\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}+\ldots.$$

5. Assume $\frac{1}{1-2x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots.$

Clearing of fractions,

$$\begin{array}{rl|rl|rl|rl|l} 1=A+B & x & +C & x^2 & +D & x^3 & +E & x^4 & +\ldots.. \\ -2A & & -2B & & -2C & & -2D & & \\ & & +A & & +B & & +C & & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} A=1 \\ B-2A=0 \\ C-2B+A=0 \\ D-2C+B=0 \\ E-2D+C=0 \end{array}\right\}\text{ whence }\left\{\begin{array}{l} A=1 \\ B=2 \\ C=3 \\ D=4 \\ E=5 \end{array}\right.$$

Substituting, we have

$$\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+\ldots.$$

6. Assume $(a^2-x^2)^{\frac{1}{2}}=A+Bx^2+Cx^4+Dx^6+\ldots..$

Squaring both members,

$$a^2 - x^2 = A^2 + \begin{array}{r|r|r|l} AB & x^2 + AC & x^4 + AD & x^6 + \ldots . \\ +AB & +B^2 & +BC & \\ & +AC & +BC & \\ & & +AD & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} A^2 = a^2 \\ 2AB = -1 \\ 2AC + B^2 = 0 \\ 2AD + 2BC = 0 \end{array}\right\} \text{whence} \left\{\begin{array}{l} A = a \\ B = -\frac{1}{2a} \\ C = -\frac{1}{8a^3} \\ D = -\frac{1}{16a^5} \end{array}\right.$$

Substituting, we have,

$$(a^2 - x^2)^{\frac{1}{2}} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \ldots .$$

7. Assume $\frac{1+x}{1-x} = A + Bx + Cx^2 + Dx^3 + \ldots ..$

Clearing of fractions,

$$1 + x = A + \begin{array}{r|r|r|l} B & x + C & x^2 + D & x^3 + \ldots .. \\ -A & -B & -C & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} A = 1 \\ B - A = 1 \\ C - B = 0 \\ D - C = 0 \end{array}\right\} \text{whence} \left\{\begin{array}{l} A = 1 \\ B = 2 \\ C = 2 \\ D = 2 \end{array}\right.$$

Substituting, we have

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \ldots ..$$

8. Assume $\frac{1}{a^3 + 6a^2x + 12ax^2 + 8x^3} = A + Bx + Cx^2$
$+ Dx^3 + \ldots ..$

Clearing of fractions,

$$\begin{array}{rr|rr|rr|l} 1 = Aa^3 + & a^3B & x + & a^3C & x^2 + & a^3D & x^3 + \ldots\ldots \\ & +6a^2A & & +\ 6a^2B & & +\ 6a^2C & \\ & & & +12aA & & +12aB & \\ & & & & & +\ 8A & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} Aa^3 = 1 \\ 6a^2A + a^3B = 0 \\ 12aA + 6a^2B + a^3C = 0 \\ 8A + 12aB + 6a^2C + a^3D = 0 \end{array}\right\} \text{whence} \left\{\begin{array}{l} A = \frac{1}{a^3} \\ B = -\frac{6}{a^4} \\ C = \frac{24}{a^5} \\ D = -\frac{80}{a^6} \end{array}\right.$$

Substituting, we have

$$\frac{1}{(a+2x)^3} = \frac{1}{a^3} - \frac{6x}{a^4} + \frac{24x^2}{a^5} - \frac{80x^3}{a^6} + \ldots\ldots$$

9. Assume $\frac{1}{a^2+2ax+x^2} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$

Clearing of fractions,

$$\begin{array}{rr|rr|rr|l} 1 = a^2A + & a^2B & x + & a^2C & x^2 + & a^2D & x^3 \\ & +2aA & & +2aB & & +2aC & \\ & & & +\ A & & +\ B & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r} a^2A = 1 \\ 2aA + a^2B = 0 \\ A + 2aB + a^2C = 0 \\ B + 2aC + a^2D = 0 \end{array}\right\} \text{whence} \left\{\begin{array}{l} A = \frac{1}{a^2} \\ B = -\frac{2}{a^3} \\ C = \frac{3}{a^4} \\ D = -\frac{4}{a^5} \end{array}\right.$$

Substituting, we have

$$\frac{1}{(a+x)^2}=\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+\ldots\ldots$$

10. Assume $\dfrac{1}{3x-x^2}=Ax^{-1}+Bx^0+Cx+Dx^2+\ldots.$

Clearing of fractions,

$$1=3A+\begin{array}{l|l|l|l}3B & x+3C & x^2+3D & x^3+\ldots\ldots\\ -A & -B & -C & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r}3A=1\\ 3B-A=0\\ 3C-B=0\\ 3D-C=0\end{array}\right\}\text{whence}\left\{\begin{array}{l}A=\frac{1}{3}\\ B=\frac{1}{9}\\ C=\frac{1}{27}\\ D=\frac{1}{81}\end{array}\right.$$

Substituting, we have,

$$\frac{1}{3x-x^2}=\frac{x^{-1}}{3}+\frac{x^0}{9}+\frac{x}{27}+\frac{x^2}{81}+\ldots\ldots$$

11. Assume $\dfrac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots\ldots$

Clearing of fractions,

$$1+2x=A+\begin{array}{l|l|l|l|l}B & x+C & x^2+D & x^3+E & x^4+\ldots\ldots\\ -A & -B & -C & -D & \\ & -A & -B & -C & \end{array}$$

Equating like coefficients,

$$\left.\begin{array}{r}A=1\\ B-A=2\\ C-B-A=0\\ D-C-B=0\\ E-D-C=0\end{array}\right\}\text{whence}\left\{\begin{array}{l}A=1\\ B=3\\ C=4\\ D=7\\ E=11\end{array}\right.$$

Substituting, we have,

$$\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+\ldots\ldots$$

12. Assume $\frac{2}{3x^2-2x^3}=Ax^{-2}+Bx^{-1}+C+Dx+Ex^2+\ldots.$

Clearing of fractions,

$$2=\begin{array}{r|r|r|r|l}3A+3B & x+3C & x^2+3D & x^3+3E & x^4+\ldots\ldots \\ -2A & -2B & -2C & -2D & \end{array}$$

Equating like coefficients,

$$\left.\begin{aligned}3A&=2\\3B-2A&=0\\3C-2B&=0\\3D-2C&=0\\3E-2D&=0\end{aligned}\right\}\text{ whence }\left\{\begin{aligned}A&=\tfrac{2}{3}\\B&=\tfrac{4}{9}\\C&=\tfrac{8}{27}\\D&=\tfrac{16}{81}\\E&=\tfrac{32}{243}\end{aligned}\right.$$

Substituting, we have,

$$\frac{2}{3x^2-2x^3}=\frac{2x^{-2}}{3}+\frac{4x^{-1}}{9}+\frac{8}{27}+\frac{16x}{81}+\frac{32x^2}{243}+\ldots\ldots$$

DECOMPOSITION OF RATIONAL FRACTIONS.

(Art. 370, pp. 294, 295.)

3. Assume $\frac{3x-5}{(x-8)(x-5)}=\frac{A}{x-8}+\frac{B}{x-5}.$

Clearing of fractions, and uniting,

$$3x-5=-5A-8B+(A+B)x$$

Equating like coefficients,

$$\left.\begin{aligned}-5A-8B&=-5\\A+\ B&=3\end{aligned}\right\}\text{ whence }\left\{\begin{aligned}A&=\tfrac{19}{3}\\B&=-\tfrac{10}{3}\end{aligned}\right.$$

Substituting, we have,

$$\frac{3x-5}{x^2-13x+40}=\frac{19}{3(x-8)}-\frac{10}{3(x-5)}$$

4. Assume $\frac{1+x^2}{x-x^3}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}.$

Clearing of fractions, and uniting,

$$1+x^2=A+(B+C)x+(B-A-C)x^2$$

Equating like coefficients,

$$\left.\begin{aligned} A&=1\\ B+C&=0\\ B-A-C&=1 \end{aligned}\right\}\text{ whence }\left\{\begin{aligned} A&=1\\ B&=1\\ C&=-1 \end{aligned}\right.$$

Substituting, we have,

$$\frac{1+x^2}{x-x^3}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}$$

5. Assume $\dfrac{x^2}{(x^2-1)(x-2)}=\dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$.

Clearing of fractions, and uniting,

$$x^2=-A-2B+2C+(-B-3C)x+(A+B+C)x^2$$

Equating like coefficients,

$$\left.\begin{aligned} -A-2B+2C&=0\\ -B-3C&=0\\ A+B+C&=1 \end{aligned}\right\}\text{ whence }\left\{\begin{aligned} A&=\tfrac{4}{3}\\ B&=-\tfrac{1}{2}\\ C&=\tfrac{1}{6} \end{aligned}\right.$$

Substituting, we have,

$$\frac{x^2}{(x^2-1)(x-2)}=\frac{4}{3(x-2)}-\frac{1}{2(x-1)}+\frac{1}{6(x+1)}$$

6. Assume $\dfrac{3x+2}{x(x+1)^3}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}+\dfrac{D}{(x+1)^3}$.

Clearing of fractions,

$$\begin{aligned} 3x+2=&\;A(x^3+3x^2+3x+1)+B(x^3+2x^2+x)\\ &+C(x^2+x)+Dx \end{aligned}$$

Arranging terms,

$$\begin{aligned} 3x+2=&\;A+(3A+B+C+D)x+(3A+2B+C)x^2\\ &+(A+B)x^3 \end{aligned}$$

Equating like coefficients,

$$\left.\begin{aligned} A&=2\\ 3A+B+C+D&=3\\ 3A+2B+C&=0\\ A+B&=0 \end{aligned}\right\}\text{ whence }\left\{\begin{aligned} A&=2\\ B&=-2\\ C&=-2\\ D&=1 \end{aligned}\right.$$

Substituting, we have,

$$\frac{3x+2}{x(x+1)^3}=\frac{2}{x}-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}$$

BINOMIAL THEOREM. ANY EXPONENT.

(ARTS. 374, 375, pp. 298–301.)

4. Substituting $2x$ for x, and $\frac{1}{2}$ for $\frac{p}{q}$, in formula (D),

$$(1+2x)^{\frac{1}{2}}=1+\tfrac{1}{2}(2x)+\frac{1(1-2)}{1\cdot2\cdot2^2}(2x)^2+\frac{1(1-2)(1-4)}{1\cdot2\cdot3\cdot2^3}(2x)^3$$

$$+\frac{1(1-2)(1-4)(1-6)}{1\cdot2\cdot3\cdot4\cdot2^4}(2x)^4+\ldots.$$

$$=1+x-\frac{1\cdot1\cdot2^2}{1\cdot2\cdot2^2}x^2+\frac{1\cdot1\cdot3\cdot2^3}{1\cdot2\cdot3\cdot2^3}x^3-\frac{1\cdot1\cdot3\cdot5\cdot2^4}{1\cdot2\cdot3\cdot4\cdot2^4}x^4+\ldots..$$

$$=1+x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3-\tfrac{5}{8}x^4+\ldots...$$

5. Substituting $-x$ for x, and 1 for n, in formula (C),

$$(1-x)^{-1}=1+1x+\frac{1(1+1)}{1\cdot2}x^2+\frac{1(1+1)(1+2)}{1\cdot2\cdot3}x^3$$

$$+\frac{1(1+1)(1+2)(1+3)}{1\cdot2\cdot3\cdot4}x^4+\ldots..$$

$$=1+x+\frac{1\cdot2}{1\cdot2}x^2+\frac{1\cdot2\cdot3}{1\cdot2\cdot3}x^3+\frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4}x^4+\ldots...$$

$$=1+x+x^2+x^3+x^4+\ldots...$$

6. Substituting $\frac{1}{5}$ for $\frac{p}{q}$ in formula (E),

$$(1+x)^{-\frac{1}{5}}=1-\tfrac{1}{5}x+\frac{1\,(1+5)}{1\cdot 2\cdot 5^2}x^2-\frac{1\,(1+5)\,(1+10)}{1\cdot 2\cdot 3\cdot 5^3}x^3+\dots\dots$$

$$=1-\tfrac{1}{5}x+\frac{1\cdot 6}{1\cdot 2\cdot 5^2}x^2-\frac{1\cdot 6\cdot 11}{1\cdot 2\cdot 3\cdot 5^3}x^3+\dots\dots$$

$$=1-\tfrac{1}{5}x+\tfrac{3}{25}x^2-\tfrac{11}{125}x^3+\dots\dots$$

7. $2\,(c+x)^{-2}=\frac{2}{c^2}\left(1+\frac{x}{c}\right)^{-2}$. Substituting in formula (C),

$$\frac{2}{c^2}\left(1+\frac{x}{c}\right)^{-2}=\frac{2}{c^2}\left(1-2\frac{x}{c}+\frac{2\,(2+1)}{1\cdot 2}\left(\frac{x}{c}\right)^2-\frac{2\,(2+1)\,(2+2)}{1\cdot 2\cdot 3}\left(\frac{x}{c}\right)^3+\dots\right)$$

$$=\frac{2}{c^2}\left(1-\frac{2x}{c}+\frac{2\cdot 3\,x^2}{1\cdot 2\,c^2}-\frac{2\cdot 3\cdot 4\,x^3}{1\cdot 2\cdot 3\,c^3}+\dots\right)$$

$$=\frac{2}{c^2}-\frac{4x}{c^3}+\frac{6x^2}{c^4}-\frac{8x^3}{c^5}+\dots$$

8. $\frac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}$. Substituting in formula (E),

$$(1+x^2)^{-\frac{1}{2}}=1-\tfrac{1}{2}x^2+\frac{1\,(1+2)}{1\cdot 2\cdot 2^2}x^4-\frac{1\,(1+2)\,(1+4)}{1\cdot 2\cdot 3\cdot 2^3}x^6+\dots\dots$$

$$=1-\tfrac{1}{2}x^2+\frac{1\cdot 3}{1\cdot 2\cdot 2^2}x^4-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 2^3}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 2^4}x^8-\dots$$

$$=1-\tfrac{1}{2}x^2+\tfrac{3}{8}x^4-\tfrac{5}{16}x^6+\tfrac{35}{128}x^8-\dots\dots$$

9. $x^2\,(x^2-y)^{-\frac{1}{2}}=x\left(1-\frac{y}{x^2}\right)^{-\frac{1}{2}}$. Substituting in formula (E),

$$x\left(1-\frac{y}{x^2}\right)^{-\frac{1}{2}}=x\left(1+\frac{1}{2}\frac{y}{x^2}+\frac{1\cdot 3\,y^2}{1\cdot 2\cdot 2^2x^4}+\frac{1\cdot 3\cdot 5\,y^3}{1\cdot 2\cdot 3\cdot 2^3x^6}+\frac{1\cdot 3\cdot 5\cdot 7\,y^4}{1\cdot 2\cdot 3\cdot 4\cdot 2^4x^8}+\dots\right)$$

$$=x\left(1+\frac{y}{2x^2}+\frac{3y^2}{8x^4}+\frac{5y^3}{16x^6}+\frac{35y^4}{128x^8}+\ldots\right)$$

$$=x+\frac{y}{2x}+\frac{3y^2}{8x^3}+\frac{5y^3}{16x^5}+\frac{35y^4}{128x^7}+\ldots\ldots$$

10. Substituting in formula (B),

$$(a^2-x^2)^{-\frac{3}{4}}=a^{-\frac{3}{2}}-(-\tfrac{3}{4})\,a^{-\frac{7}{2}}x^2+\frac{(-\frac{3}{4})(-\frac{7}{4})}{1\,.\,2}a^{-\frac{11}{2}}x^4$$
$$-\frac{(-\frac{3}{4})(-\frac{7}{4})(-\frac{11}{4})}{1\,.\,2\,.\,3}a^{-\frac{15}{2}}x^6+\ldots\ldots$$

$$=a^{-\frac{3}{2}}+\tfrac{3}{4}a^{-\frac{7}{2}}x^2+\frac{3\,.\,7}{1\,.\,2\,.\,4^2}a^{-\frac{11}{2}}x^4+\frac{3\,.\,7\,.\,11}{1\,.\,2\,.\,3\,.\,4^3}a^{-\frac{15}{2}}x^6+\ldots\ldots$$

$$=a^{-\frac{3}{2}}+\tfrac{3}{4}a^{-\frac{7}{2}}x^2+\tfrac{21}{32}a^{-\frac{11}{2}}x^4+\tfrac{77}{128}a^{-\frac{15}{2}}x^6+\ldots\ldots$$

11. $\dfrac{m}{\sqrt{b^2+c^4}}=m\,(b^2+c^4)^{-\frac{1}{2}}=\dfrac{m}{b}\left(1+\dfrac{c^4}{b^2}\right)^{-\frac{1}{2}}$. Substituting in formula (E),

$$\frac{m}{b}\left(1+\frac{c^4}{b^2}\right)^{-\frac{1}{2}}=\frac{m}{b}\left(1-\frac{1}{2}\frac{c^4}{b^2}+\frac{1\,.\,3}{1\,.\,2\,.\,2^2}\frac{c^8}{b^4}-\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3\,.\,2^3}\frac{c^{12}}{b^6}\right.$$
$$\left.+\frac{1\,.\,3\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,4\,.\,2^4}\frac{c^{16}}{b^8}-\ldots\right)$$

$$=\frac{m}{b}\left(1-\frac{c^4}{2b^2}+\frac{3c^8}{8b^4}-\frac{5c^{12}}{16b^6}+\frac{35c^{16}}{128b^8}-\ldots\right)$$

13. $\sqrt[3]{9}=\sqrt[3]{8+1}=2\,(1+\tfrac{1}{8})^{\frac{1}{3}}$

$$=2\left(1+\frac{1}{3}\cdot\frac{1}{8}+\frac{1\,(1-3)}{1\,.\,2\,.\,3^2}\left(\frac{1}{8}\right)^2+\frac{1\,(1-3)\,(1-6)}{1\,.\,2\,.\,3\,.\,3^3}\left(\frac{1}{8}\right)^3+\ldots\ldots\right)$$

$$=2\left(1+\frac{1}{3\,.\,8}-\frac{1\,.\,2}{1\,.\,2\,.\,3^2\,.\,8^2}+\frac{1\,.\,2\,.\,5}{1\,.\,2\,.\,3\,.\,3^3\,.\,8^3}\right.$$
$$\left.-\frac{1\,.\,2\,.\,5\,.\,8}{1\,.\,2\,.\,3\,.\,4\,.\,3^4\,.\,8^4}+\ldots\ldots\right)$$

$$= 2 + \frac{1}{3 \,.\, 2^2} - \frac{1}{3^2 \,.\, 2^5} + \frac{5}{3^4 \,.\, 2^8} - \frac{5}{3^5 \,.\, 2^{10}} + \ldots\ldots$$

$$= 2 + \tfrac{1}{12} - \tfrac{1}{288} + \tfrac{5}{20736} - \tfrac{5}{248832} + \ldots\ldots$$

$$= 2 + .08333 - .00347 + .00024 - .00002$$

$$= 2.08357 - .00349 = 2.08008$$

14. $\sqrt[5]{39} = \sqrt[5]{32 + 7} = 2\,(1 + \tfrac{7}{32})^{\frac{1}{5}}$

$$= 2\left(1 + \tfrac{1}{5} \cdot \tfrac{7}{32} + \frac{1\,(1-5)}{1 \,.\, 2 \,.\, 5^2}\,(\tfrac{7}{32})^2 + \frac{1\,(1-5)\,(1-10)}{1 \,.\, 2 \,.\, 3 \,.\, 5^3}\,(\tfrac{7}{32})^3 + \ldots\ldots\right)$$

$$= 2\left(1 + \frac{7}{5 \,.\, 2^5} - \frac{1 \,.\, 4 \,.\, 7^2}{1 \,.\, 2 \,.\, 5^2 \,.\, 2^{10}} + \frac{1 \,.\, 4 \,.\, 9 \,.\, 7^3}{1 \,.\, 2 \,.\, 3 \,.\, 5^3 \,.\, 2^{15}} - \frac{1 \,.\, 4 \,.\, 9 \,.\, 14 \,.\, 7^4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5^4 \,.\, 2^{20}} + \ldots\ldots\right)$$

$$= 2 + \frac{7}{5 \,.\, 2^4} - \frac{7^2}{5^2 \,.\, 2^8} + \frac{3 \,.\, 7^3}{5^3 \,.\, 2^{13}} - \frac{3 \,.\, 7^5}{5^4 \,.\, 2^{19}} + \ldots\ldots$$

$$= 2 + \tfrac{7}{80} - \tfrac{49}{6400} + \tfrac{1029}{1024000} - \tfrac{50421}{327680000} + \ldots\ldots$$

$$= 2 + .0875 - .00766 + .001. - .00015$$

$$= 2.0885 - .0078 = 2.0807$$

15. $\sqrt[5]{260} = \sqrt[5]{243 + 17} = 3\left(1 + \frac{17}{3^5}\right)^{\frac{1}{5}}$

$$= 3\left(1 + \frac{1 \,.\, 17}{5 \,.\, 3^5} - \frac{1 \,.\, 4 \,.\, 17^2}{1 \,.\, 2 \,.\, 5^2 \,.\, 3^{10}} + \frac{1 \,.\, 4 \,.\, 9 \,.\, 17^3}{1 \,.\, 2 \,.\, 3 \,.\, 5^3 \,.\, 3^{15}} - \ldots\ldots\right)$$

$$= 3 + \frac{17}{5 \,.\, 3^4} - \frac{2 \,.\, 17^2}{5^2 \,.\, 3^9} + \frac{2 \,.\, 17^3}{5^3 \,.\, 3^{13}} - \ldots\ldots$$

$$= 3 + \tfrac{17}{405} - \tfrac{578}{492075} + \tfrac{9826}{199290375} - \ldots\ldots$$

$$= 3 + .041975 - .001175 + .00005$$

$$= 3.042025 - .001175 = 3.04085$$

16. $\sqrt[7]{108} = \sqrt[7]{128 - 20} = 2\left(1 - \frac{5}{32}\right)^{\frac{1}{7}}$

$$= 2\left(1 - \frac{1}{7} \cdot \frac{5}{32} + \frac{1(1-7)}{1 \,.\, 2 \,.\, 7^2}\left(\frac{5}{32}\right)^2 - \frac{1(1-7)(1-14)}{1 \,.\, 2 \,.\, 3 \,.\, 7^3}\left(\frac{5}{32}\right)^3 + \ldots\right)$$

$$= 2\left(1 - \frac{5}{7 \,.\, 2^5} - \frac{1 \,.\, 6 \,.\, 5^2}{1 \,.\, 2 \,.\, 7^2 \,.\, 2^{10}} - \frac{1 \,.\, 6 \,.\, 13 \,.\, 5^3}{1 \,.\, 2 \,.\, 3 \,.\, 7^3 \,.\, 2^{15}}\right.$$

$$\left. - \frac{1 \,.\, 6 \,.\, 13 \,.\, 20 \,.\, 5^4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 7^4 \,.\, 2^{20}} - \ldots\ldots\right)$$

$$= 2 - \frac{5}{7 \,.\, 2^4} - \frac{3 \,.\, 5^2}{7^2 \,.\, 2^9} - \frac{13 \,.\, 5^3}{7^3 \,.\, 2^{14}} - \frac{13 \,.\, 5^5}{7^4 \,.\, 2^{19}} - \frac{13 \,.\, 3^3 \,.\, 5^5}{7^5 \,.\, 2^{24}} - \ldots\ldots$$

$$= 2 - \tfrac{5}{112} - \tfrac{75}{25088} - \tfrac{1625}{5619712} - \tfrac{40625}{1258815488} - \ldots\ldots$$

$$= 2 - .044643 - .002989 - .000289 - .000032 - .000004$$

$$= 2 - .047957 = 1.952043$$

REVERSION OF SERIES.

(ART. 376, pp. 301-303.)

1. Here $a, b, c, \ldots\ldots$ are all 1. Substituting this value in formula (A), we obtain

$$x = y - y^2 + y^3 - y^4 + \ldots\ldots$$

2. Here $a = 2$, $b = 3$, $c = 4$, $d = 5$, etc. Substituting these values in formula (B), we obtain

$$x = \tfrac{1}{2}y - \tfrac{3}{16}y^3 + \tfrac{19}{128}y^5 - \tfrac{152}{1024}y^7 + \ldots$$

3. Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, $d = \frac{1}{24}$, etc. Substituting these values in formula (A), and using $y - 1$ for y, we have for the value of x,

$$(y-1) - \tfrac{1}{2}(y-1)^2 + (\tfrac{1}{2} - \tfrac{1}{6})(y-1)^3 - (\tfrac{5}{8} - \tfrac{5}{12} + \tfrac{1}{24})(y-1)^4 + \ldots$$

Or, $x = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \ldots$

4. Here $a=1$, $b=-\frac{1}{3}$, $c=\frac{1}{5}$, $d=-\frac{1}{7}$, etc. Substituting these values in formula (B), we have

$$x=y-(-\tfrac{1}{3})\,y^3+(\tfrac{1}{3}-\tfrac{1}{5})\,y^5-(-\tfrac{4}{9}+\tfrac{8}{15}-\tfrac{1}{7})\,y^7+\ldots.$$

$$\text{Or,}\quad x=y+\tfrac{1}{3}y^3+\tfrac{2}{15}y^5+\tfrac{17}{315}y^7+\ldots..$$

5. Assume $y=Ax+Bx^2+Cx^3+\ldots..$ Substituting this value of y in the original equation, we have

$$ax+bx^2+cx^3+\ldots.=mAx+\left.\begin{array}{r} mB \\ nA^2 \end{array}\right|x^2+\left.\begin{array}{r} mC \\ +2nAB \\ +\ pA^3 \end{array}\right|x^3+\ldots..$$

Equating the like coefficients of x,

$$\left.\begin{array}{r} mA=a \\[2ex] mB+nA^2=b \\[2ex] mC+2nAB+pA^3=c \end{array}\right\}\text{ whence }\left\{\begin{array}{l} A=\dfrac{a}{m} \\[2ex] B=\dfrac{bm^2-a^2n}{m^3} \\[2ex] C=\dfrac{cm^4-a^3mp-2an(bm^2-a^2n)}{m^5} \end{array}\right.$$

Substituting, we have

$$y=\frac{a}{m}x+\frac{bm^2-a^2n}{m^3}x^2+\frac{cm^4-a^3mp-2an(bm^2-a^2n)}{m^5}x^3+\ldots...$$

SUMMATION OF INFINITE SERIES.

RECURRING SERIES.

(Art. 381, pp. 305-307.)

2. Here, $a=1$, $b=2$, $c=3$, $d=5$, etc. Substituting these values in formulas (A),

$$p=\frac{5-6}{3-4}=1,\quad q=\frac{9-10}{3-4}=1;$$

whence, by formula (D),

$$S=\frac{1+(2-1)x}{1-x-x^2}=\frac{1+x}{1-x-x^2}$$

3. Here, $a = \frac{a}{b}$, $b = -\frac{ac}{b^2}$, $c = \frac{ac^2}{b^3}$, $d = -\frac{ac^3}{b^4}$, etc. It is evident that each of these is the product of the next preceding one by $-\frac{c}{b}$; hence the series is a geometrical progression, or a recurring series of the first order, in which $p = -\frac{c}{b}$. Substituting in formula (E),

$$S = \frac{\frac{a}{b}}{1 + \frac{cx}{b}} = \frac{a}{b + cx}$$

4. Here, $a = 4$, $b = 9$, $c = 21$, $d = 51$, etc. Substituting in formulas (A),

$$p = \frac{204 - 189}{84 - 81} = \frac{15}{3} = 5$$

$$q = \frac{441 - 459}{84 - 81} = \frac{-18}{3} = -6.$$

Whence, by formula (D),

$$S = \frac{4 + (9 - 20)x}{1 - 5x + 6x^2} = \frac{4 - 11x}{1 - 5x + 6x^2}$$

5. Here $a = 1$, $b = 3$, $c = 5$, $d = 7$, etc. Substituting in formulas (A),

$$p = \frac{7 - 15}{5 - 9} = \frac{-8}{-4} = 2$$

$$q = \frac{25 - 21}{5 - 9} = \frac{4}{-4} = -1$$

Whence, by formula (D),

$$S = \frac{1 + (3 - 2)x}{1 - 2x + x^2} = \frac{1 + x}{(1 - x)^2}$$

6. Here $a = 2$, $b = -1$, $c = 2$, $d = -5$, $e = 10$, $f = -17$, etc. Substituting in formulas (B),

$$-5 = 2p - q + 2r \qquad (1)$$

$$10 = -5p + 2q - r \qquad (2)$$

$$-17 = 10p - 5q + 2r \qquad (3)$$

Multiplying (2) by 2, $20 = -10p + 4q - 2r$ (4)

Adding (3) and (4), $q = -3$ (5)

Adding (1) and (4), $15 = -8p + 3q$ (6)

Subst. (5) in (6), $8p = -24$

Whence, $p = -3$

Subst. in (2), $r = -10 + 15 - 6 = -1$

Substituting in formula (F), and writing a for x,

$$S = \frac{2 + (-1 + 6)a + (2 - 3 + 6)a^2}{1 + 3a + 3a^2 + a^3} = \frac{2 + 5a + 5a^2}{(1 + a)^3}$$

7. Here $a = 3$, $b = 5$, $c = 7$, $d = 13$, $e = 23$, $f = 45$, etc
Substituting in formulas (B),

$$13 = 7p + 5q + 3r \qquad (1)$$

$$23 = 13p + 7q + 5r \qquad (2)$$

$$45 = 23p + 13q + 7r \qquad (3)$$

Multiplying (1) by 5, $65 = 35p + 25q + 15r$ (4)

Multiplying (2) by 3, $69 = 39p + 21q + 15r$ (5)

Subtracting (4) from (5), $4 = 4p - 4q$ (6)

Multiplying (2) by 7, $161 = 91p + 49q + 35r$ (7)

Multiplying (3) by 5, $225 = 115p + 65q + 35r$ (8)

Subtracting (7) from (8), $64 = 24p + 16q$ (9)

Multiplying (6) by 4, $16 = 16p - 16q$ (10)

Adding (9) and (10), $80 = 40p$

Whence, $p = 2$

Substituting in (6), $q = 1$

Substituting in (1), $3r = 13 - 14 - 5 = -6$

Whence, $r = -2$

Substituting in formula (F),

$$S = \frac{3 + (5-6)x + (7-10-3)x^2}{1-2x-x^2+2x^3} = \frac{3-x-6x^2}{1-2x-x^2+2x^3}$$

DIFFERENTIAL METHOD.

(ART. 385, pp. 309–311.)

3.

$$\begin{array}{c}
6,\ 9,\ 17,\ 35,\ 63,\ 99,\ \ldots\ldots \\
3,\ 8,\ 18,\ 28,\ 36,\ \ldots\ldots \\
5,\ 10,\ 10,\ 8,\ \ldots\ldots \\
5,\ 0,\ -2,\ \ldots\ldots \\
-5,\ -2,\ \ldots\ldots \\
3,\ \ldots\ldots
\end{array}$$

4.

$$\begin{array}{c}
3,\ 6,\ 11,\ 17,\ 24,\ 36,\ 50,\ \ldots\ldots \\
3,\ 5,\ 6,\ 7,\ 12,\ 14,\ \ldots\ldots \\
2,\ 1,\ 1,\ 5,\ 2,\ \ldots\ldots \\
-1,\ 0,\ 4,\ -3,\ \ldots\ldots \\
1,\ 4,\ -7,\ \ldots\ldots \\
3,\ -11,\ \ldots\ldots \\
-14,\ \ldots\ldots
\end{array}$$

5.

$$\begin{array}{c}
3,\ 5,\ 8,\ 12,\ 17,\ \ldots\ldots \\
2,\ 3,\ 4,\ 5,\ \ldots\ldots \\
1,\ 1,\ 1,\ \ldots\ldots \\
0,\ 0,\ \ldots\ldots
\end{array}$$

Hence $a = 3$, $d_1 = 2$, $d_2 = 1$, $d_3 = 0$, and $n = 7$.
Substituting in formula (A),

$$T_7 = 3 + 12 + \frac{6 \cdot 5}{1 \cdot 2} = 3 + 12 + 15 = 30$$

6. 1, 4, 10, 20, 35, ...
3, 6, 10, 15, ...
3, 4, 5, ...
1, 1, ...
0, ...

Hence $a = 1$, $d_1 = 3$, $d_2 = 3$, $d_3 = 1$, $d_4 = 0$, and $n = 12$. Substituting in formula (B),

$$S_{12} = 12 + \frac{12 \cdot 11}{1 \cdot 2} 3 + \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} 3 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}$$
$$= 12 + 198 + 660 + 495 = 1365$$

7. 1, 2, 3, 4, 5, ...
1, 1, 1, 1, ...
0, 0, 0, ...

Hence $a = 1$, $d_1 = 1$, $d_2 = 0$, and $n = 100$. Substituting in formula (B),

$$S_{100} = 100 + \frac{100 \cdot 99}{1 \cdot 2} = 100 + 4950 = 5050$$

8. 1, 4, 9, 16, 25, ...
3, 5, 7, 9, ...
2, 2, 2, ...
0, 0, ...

Hence $a = 1$, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, and $n = 15$. Substituting in formula (A),

$$T_{15} = 1 + 14 \times 3 + \frac{14 \cdot 13}{1 \cdot 2} 2 = 1 + 42 + 182 = 225$$

9. 1, 8, 27, 64, 125, ...
7, 19, 37, 61, ...
12, 18, 24, ...
6, 6, ...
0, ...

Hence $a = 1$, $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, and $d_4 = 0$. Substituting in formula (B),

$$S_n = n + \frac{n(n-1)}{1 . 2} 7 + \frac{n(n-1)(n-2)}{1 . 2 . 3} 12 + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4} 6$$

$$= n + \tfrac{7}{2}n^2 - \tfrac{7}{2}n + 2n^3 - 6n^2 + 4n + \tfrac{1}{4}n^4 - \tfrac{3}{2}n^3 + \tfrac{11}{4}n^2 - \tfrac{3}{2}n = \tfrac{1}{4}n^2 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^4 = \tfrac{1}{4}(n^4 + 2n^3 + n^2)$$

10. 1, 16, 81, 256, 625, 1296, ...
15, 65, 175, 369, 671, ...
50, 110, 194, 302, ...
60, 84, 108, ...
24, 24, ...
0, ...

Hence, $a = 1$, $d_1 = 15$, $d_2 = 50$, $d_3 = 60$, $d_4 = 24$, and $d_5 = 0$. Substituting in formula (B),

$$S_n = n + \frac{n(n-1)}{1 . 2} 15 + \frac{n(n-1)(n-2)}{1 . 2 . 3} 50 + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4} 60 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 . 2 . 3 . 4 . 5} 24$$

$$= n + \tfrac{15}{2}n^2 - \tfrac{15}{2}n + \tfrac{25}{3}n^3 - 25n^2 + \tfrac{50}{3}n + \tfrac{5}{2}n^4 - 15n^3 + \tfrac{55}{2}n^2 - 15n + \tfrac{1}{5}n^5 - 2n^4 + 7n^3 - 10n^2 + \tfrac{24}{5}n$$

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

11. 1, 3, 6, 10, 15, ...
2, 3, 4, 5, ...
1, 1, 1, ...
0, 0, ...

Hence $a = 1$, $d_1 = 2$, $d_2 = 1$, $d_3 = 0$, and $n = 9$. Substituting in formula (B),

$$S_9 = 9 + \frac{9 . 8}{1 . 2} 2 + \frac{9 . 8 . 7}{1 . 2 . 3} = 9 + 72 + 84 = 165$$

12. The series is the same as in the 8th example; hence $a=1$, $d_1=3$, $d_2=2$, $d_3=0$, and $n=25$. Substituting in formula (B),

$$S_{25}=25+\frac{25\,.\,24}{1\,.\,2}3+\frac{25\,.\,24\,.\,23}{1\,.\,2\,.\,3}2=25+900+4600=5525.$$

INTERPOLATION.

(Art. 387, pp. 311–313.)

3. First differences, .0207, .0205, .0203

Second differences, —.0002, —.0002

Hence $a=4$, $d_1=.0207$, $d_2=-.0002$, and $p=2.5$. Substituting in the formula,

$$t=4+2.5\times .0207+\frac{2.5\times 1.5}{1.2}(-.0002)$$
$$=4+.0518-.0004=4.0514$$

4. First differences, —.69, —.71, —.72

Second differences, —.02, —.01

Third difference, .01

Hence, $a=45.28$, $d_1=-.69$, $d_2=-.02$, $d_3=.01$, and $p=1\frac{1}{2}$. Substituting in the formula,

$$t=45.28+\tfrac{3}{2}(-.69)+\frac{\frac{3}{2}\times\frac{1}{2}}{1\,.\,2}(-.02)$$
$$=45.28-1.035-.008=44.24$$

The fourth term would be less than .01, for $d_3=.01$.

5. First differences, .051933, .050480, .049124

Second differences, —.001453, —.001356

Third difference, .000097

Hence, $a=3.556893$, $d_1=.051933$, $d_2=-.001453$, $d_3=.000097$, and $p=1\frac{1}{2}$. Substituting in the formula,

$$\begin{aligned} t &= 3.556893 + \tfrac{3}{2}(.051933) + \tfrac{3}{8}(-.001453) - \tfrac{1}{16}(.000097) \\ &= 3.556893 + .077899 - .000545 - .000006 = 3.634241 \end{aligned}$$

6. First differences, .080, .086, .092

Second differences, .006, .006

Hence $a = 1.145$, $d_1 = .080$, $d_2 = .006$, and $p = 2\tfrac{1}{2}$. Substituting in the formula,

$$\begin{aligned} t &= 1.145 + \tfrac{5}{2}(.080) + \frac{\frac{5}{2} \times \frac{3}{2}}{1\,.\,2}(.006) \\ &= 1.145 + .200 + .011 = 1.356 \end{aligned}$$

If the work were carried out to four places of decimals, we should obtain 1.357.

LOGARITHMS.

TABLES OF COMMON LOGARITHMS.

(Art. 417, pp. 326, 327.)

1. Obtained directly from Table II.

2.
$$\begin{array}{lrl} \log & 49 = & 1.690196 \\ \log & 15 = & 1.176091 \\ \hline \log & 735 = & 2.866287 \\ \log & 10000 = & 4 \\ \hline \log & .0735 = & \bar{2}.866287 \end{array}$$

3. $\log 1728 = 3 \log 12 = 3 \times 1.079181 = 3.237543.$

4. Obtained directly from Table I., by Art. 395.

(5.)

$$\begin{aligned}\log 1.17 &= 0.068186\\ \log 2 &= \underline{0.301030}\\ \log 2.34 &= 0.369216\end{aligned}$$

(6.)

$$\begin{aligned}\log 37 &= 1.568202\\ \log 94 &= \underline{1.973128}\\ \log \tfrac{37}{94} &= \bar{1}.595074\end{aligned}$$

(7.)

$$\begin{aligned}\log 3 &= 0.477121\\ \log 107 &= \underline{2.029384}\\ & \ \bar{2}.447737\end{aligned}$$

(8.)

$$\begin{aligned}\log 81 &= 1.908485\\ \log 5 &= \underline{0.698970}\\ & \ 2.607455\\ \log 23 &= \underline{1.361728}\\ \log 17\tfrac{14}{23} &= 1.245727\end{aligned}$$

(9.)

$$\begin{aligned}&\underline{\log 1038 - \log 1037 = 418}\\ \log 1.037 &= 0.015779\\ 418 \times .5 &= \underline{209}\\ \log 1.0375 &= 0.015988\end{aligned}$$

(10.)

$$\begin{aligned}&\underline{\log 1049 - \log 1048 = 414.}\\ \log 104.8 &= 2.020361\\ 414 \times .57 &= \underline{236}\\ & \ 2.020597\end{aligned}$$

12. $3.010724 = \log 1025.$

(13.)

$$\begin{aligned}\text{Given } \log &= 0.009663\\ \log 1.022 &= \underline{0.009451}\\ & \ 212\\ 212 \div 424 &= .5\end{aligned}$$

Hence $0.009663 = \log 1.0225$

(14.)

$$\begin{aligned}\text{Given } \log &= 3.009264\\ \log 1021 &= \underline{3.009026}\\ & \ 238\\ 238 \div 424 &= .56\end{aligned}$$

Hence $3.009264 = \log 1021.56$

EXPONENTIAL EQUATIONS.

(ART. 420, pp. 328, 329.)

2. Given $3^x = 15$

Applying log, $x \log 3 = \log 15$

Whence, $x = \dfrac{\log 15}{\log 3} = \dfrac{1.17609}{0.47712} = 2.465$

3. Given $5^x = 100$

Appl. log, $x \log 5 = \log 100$

Whence, $x = \dfrac{\log 100}{\log 5} = \dfrac{2}{.699} = 2.861$

4. Given $a b^x = c$

Appl. log, $\log a + x \log b = \log c$

Or, $x \log b = \log c - \log a$

Whence, $x = \dfrac{\log c - \log a}{\log b}$

6. Given $6^{\frac{2}{x}} = \dfrac{5}{2}$

Raising to xth power, $6^2 = (\frac{5}{2})^x$

Appl. log, $\log 36 = x (\log 5 - \log 2)$

Whence, $x = \dfrac{\log 36}{\log 5 - \log 2} = \dfrac{1.5563}{.699 - .301}$

Or, $x = \dfrac{1.5563}{.3979} = 3.911$

8. Given $\dfrac{a b^x + c}{d} = n$

Whence, $a b^x = d n - c$

Appl. log, $\log a + x \log b = \log (d n - c)$

Whence, $x = \dfrac{\log (d n - c) - \log a}{\log b}$

9. Given $a r^{n-1} = l$

Appl. log, $\log a + (n - 1) \log r = \log l$

Whence, $n - 1 = \dfrac{\log l - \log a}{\log r}$

Or, $n = \dfrac{\log l - \log a}{\log r} + 1$

10. Given $$a(r^n - 1) = (r - 1)S$$

Or, $$ar^n = a + (r - 1)S$$

Appl. log, $$\log a + n \log r = \log[a + (r - 1)S]$$

Whence, $$n = \frac{\log[a + (r - 1)S] - \log a}{\log r}$$

11. Eliminating r from the two fundamental equations we obtain, as in the Table, Art. 344,

$$l(S - l)^{n-1} = a(S - a)^{n-1}$$

Appl. log,

$$\log l + (n - 1)\log(S - l) = \log a + (n - 1)\log(S - a)$$

Or, $$(n - 1)\log(S - a) - (n - 1)\log(S - l) = \log l - \log a$$

Whence, $$n - 1 = \frac{\log l - \log a}{\log(S - a) - \log(S - l)}$$

Or, $$n = \frac{\log l - \log a}{\log(S - a) - \log(S - l)} + 1$$

12. Eliminating a from the two fundamental equations we obtain, as in the Table,

$$lr^n - (r - 1)Sr^{n-1} = l$$

Factoring, $$[lr - (r - 1)S]r^{n-1} = l$$

Appl. log, $$\log[lr - (r - 1)S] + (n - 1)\log r = \log l$$

Whence, $$n - 1 = \frac{\log l - \log[lr - (r - 1)S]}{\log r}$$

Or, $$n = \frac{\log l - \log[lr - (r - 1)S]}{\log r} + 1$$

(Art. 421, pp. 329, 330.)

2. Here, $x \log x = \log 10 \doteq 1$, and upon trial the value of x is found to be between 2 and 3.

$$x \log x = 2 \log 2 = 2 \times 0.301 = 0.60$$
$$x \log x = 3 \log 3 = 3 \times 0.477 = \underline{1.43}$$
$$\text{Difference of results} = \ .83$$

$$.83 : 1 :: .43 : .5, \text{ the correction.}$$

$$x \log x = 2.5 \log 2.5 = 2.5 \times 0.39794 = 0.99485$$
$$x \log x = 2.6 \log 2.6 = 2.6 \times 0.41497 = \underline{1.07892}$$
$$\text{Difference of results} = 0.08407$$
$$.08407 : 0.1 :: .00515 : .006, \text{ the correction.}$$
$$x = 2.5 + .006 = 2.506$$

3. The value of x is found by trial to be exactly 4.

4. Here $x \log x = \log 5 = 0.699$, and upon trial the value of x is found to be between 2 and 3. As in the 2d example, the difference of the results is found to be .83.

$$.83 : 1 :: 0.1 : .1, \text{ the correction.}$$
$$x \log x = 2.1 \log 2.1 = 2.1 \times 0.32222 = 0.6767$$
$$x \log x = 2.2 \log 2.2 = 2.2 \times 0.34242 = \underline{0.7533}$$
$$\text{Difference of results} = .0766$$
$$.0766 : 0.1 :: .0223 : .029, \text{ the correction.}$$
$$x = 2.1 + .029 = 2.129$$

COMPOUND INTEREST AND ANNUITIES.

(Art. 428, pp. 332–334.)

1. Given $$A = \frac{a[(1+r)^n - 1]}{r}$$

Or, $$a(1+r)^n = a + rA$$

Appl. log, $$\log a + n \log(1+r) = \log(a+rA)$$

Whence, $$n = \frac{\log(a+rA) - \log a}{\log(1+r)}$$

2. Given $$P = \frac{a[(1+r)^n - 1]}{r(1+r)^n}$$

Clear. of fract., $$rP(1+r)^n = a(1+r)^n - a$$

Factoring, etc., $$(a - rP)(1+r)^n = a$$

Appl. log, $$\log(a - rP) + n \log(1+r) = \log a$$

Whence, $$n = \frac{\log a - \log(a - rP)}{\log(1+r)}$$

3. Here $P = 1$, $A = 2$, and $r = .06$. Substituting in formula (4), Art. 423,

$$n = \frac{\log 2 - \log 1}{\log 1.06} = \frac{0.3010}{0.0253} = 11.9$$

4. Here $P = 16$, $r = .05$, and $n = 30$. Substituting in formula (1), Art. 423, $\log A = \log 16 + 30 \log 1.05$.

$$\begin{array}{rlcrl}
\log 16 = & 1.204120 & \qquad & \log 70 = & 1.845098 \\
30 \log 1.05 = & \underline{0.635670} & & \log 69 = & \underline{1.838849} \\
 & 1.839790 & & & 6249 \\
\log 69 = & \underline{1.838849} & & 941 \div 6249 = & .15 \\
 & 941 & & 69 + .15 = & 69.15
\end{array}$$

5. Here $P = 500$, $r = .03$. and $n = 18$. Substituting in formula (1), Art. 423, or in formula (B),

$$\log A = \log 500 + 18 \log 1.03.$$

$$\begin{array}{rl}
\log 500 = & 2.698970 \\
18 \log 1.03 = & \underline{0.231066} \\
\log 851.21 = & 2.930036
\end{array}$$

6. Here $P = 2233.57$, $r = .06$, and $A = 4000$. Substituting in formula (4), Art. 423

$$n = \frac{3.602060 - 3.349000}{0.025306} = \frac{.253060}{.025306} = 10$$

7.

$$\begin{array}{rl}
\log A = & \log 4000 = 3.602060 \\
n \log (1 + r) = & 10 \log 1.06 = \underline{0.253060} \\
\log P = & \log 2233.57 = 3.349000
\end{array}$$

8.

$$\begin{array}{rl}
\log A = & \log 4000 = 3.602060 \\
\log P = & \log 2233.57 = \underline{3.349000} \\
 & 10\overline{)\,0.253060} \\
\log (1 + r) = & \log 1.06 = 0.025306 \\
r = & 1.06 - 1 = .06
\end{array}$$

9. Here $P=1$, $A=2$, and $r=.20$. Substituting in formula (4), Art. 423,

$$n=\frac{\log 2-\log 1}{\log 1.2}=\frac{0.301}{0.079}=3.8$$

$$3.8\times 10 \text{ years}=38 \text{ years}.$$

10. Here $a=500$, $n=10$, and $r=.06$. Substituting in formula (D),

$$P=\frac{500\,(1.06^{10}-1)}{.06\times 1.06^{10}}$$

$$\begin{array}{rl}\log 1.06 = & 0.025306\\ & \underline{\qquad\quad 10}\\ \log 1.79085 = & 0.253060\end{array}$$

$$\begin{array}{rl}\log .06 = & \bar{2}.778151\\ 10\log 1.06 = & \underline{0.253060}\\ \log(.06\times 1.06^{10}) = & \bar{1}.031211\\ \log .79085 = & \underline{\bar{1}.898095}\\ & 0.866884\\ \log 500 = & \underline{2.698970}\\ \log 3680.04 = & 3.565854\end{array}$$

11. Here $a=1200$ and $r=.05$. Substituting in formula (E),

$$P=\frac{1200}{.05}=24000.$$

12. Here $r=.06$, $n=10$, and $P=3680.04$. Formula (D) may be put in the form

$$a=\frac{rP(1+r)^n}{(1+r)^n-1}$$

$$\begin{array}{rrl}\log r = & \log .06 = & \bar{2}.778151\\ \log P = & \log 3680.04 = & 3.565854\\ \log(1+r)^n = & 10\ \log 1.06 = & \underline{0.253060}\\ & & 2.597065\\ \text{From Prob. 10, } \log[(1+r)^n-1] = & & \underline{\bar{1}.898095}\\ \log a = & \log 500 = & 2.698970\end{array}$$

13. Here $P=8000$, $r=.05$, and $a=850$. Substituting these values in the formula of Prob. 2,

$$n = \frac{\log 850 - \log 450}{\log 1.05} = \frac{2.9294 - 2.6532}{.02119} = \frac{.2762}{.02119} = 13.034$$

$$= 13 \text{ years } 12 \text{ days.}$$

14. Here $a = 150$, $r = .015$, and $n = 22$. Substituting in formula (C),

$$A = \frac{150\,(1.015^{22} - 1)}{.015}$$

$$\begin{array}{rl} \log 1.015 = & 0.006466 \\ & \underline{22} \\ \log 1.38756 = & 0.142252 \end{array} \qquad\qquad \frac{150 \times .38756}{.015} = 3875.6$$

GENERAL THEORY OF EQUATIONS.

GENERAL PROPERTIES.

DIVISIBILITY OF EQUATIONS.

(Art. 435, p. 336.)

3. $(x^3 + x^2 - 17x + 15) \div (x - 1) = x^2 + 2x - 15$, with no remainder.

4. Ans. $x + 2$. 5. Ans. -3.

NUMBER OF ROOTS.

(Art. 439, pp. 338, 339.)

2. Dividing $x^3 - 19x + 30$ by $x - 2$, we obtain $x^2 + 2x - 15$.

Depressed equation, $x^2 + 2x - 15 = 0$

Completing square, $x^2 + 2x + 1 = 16$

Evolving, $x + 1 = \pm 4$

Whence, $x = 3$, and -5

3. It is evident that one root is a. Dividing $x^3 - a^3$ by $x - a$, we obtain $x^2 + ax + a^2$.

Depressed equation, $x^2 + ax + a^2 = 0$

Completing square, $x^2 + ax + \frac{a^2}{4} = -\frac{3a^2}{4}$

Evolving, $x + \frac{a}{2} = \pm \frac{a}{2}\sqrt{-3}$

Whence, $x = \frac{a}{2}(-1 \pm \sqrt{-3})$

4. Dividing $x^3 + x^2 - 16x + 20$ by $x + 5$, we obtain $x^2 - 4x + 4$.

Depressed equation, $x^2 - 4x + 4 = 0$

Evolving, $x - 2 = 0$

Whence, $x = 2$, and 2

5. Dividing $x^4 - 3x^3 - 14x^2 + 48x - 32$ by $x - 1$ and $x - 2$, or by $x^2 - 3x + 2$, we obtain $x^2 - 16$; hence the depressed equation is $x^2 - 16 = 0$, or $x^2 = 16$, whence $x = \pm 4$.

6. Dividing $x^4 - 7x^3 + 3x + 3$ by $x - 1$, we obtain $x^3 - 6x^2 - 6x - 3$; hence the depressed equation is

Ans. $x^3 - 6x^2 - 6x - 3 = 0$.

FORMATION OF EQUATIONS.

(Art. 440, pp. 339, 340.)

2. $(x+1)(x+3)(x+5) = x^3 + 9x^2 + 23x + 15$.

3. $(x-1)(x-2)(x-3)(x-4)$
$= x^4 - 10x^3 + 35x^2 - 50x + 24$.

4. $(x-1)(x-\frac{1}{2})(x-\frac{1}{3}) = x^3 - \frac{11}{6}x^2 + x - \frac{1}{6}$.

5. $(x-4)(x-4)(x-5) = x^3 - 13x^2 + 56x - 80$.

6. $(x-5)(x+2)(x+3) = x^3 - 19x - 30$.

7. $(x-1)(x+1)(x-2)(x+2) = (x^2-1)(x^2-4)$
$= x^4 - 5x^2 + 4.$

8. $(x-0)(x+1)(x-3)(x-4) = x^4 - 6x^3 + 5x^2 + 12x.$

COMPOSITION OF COEFFICIENTS.

(ART. 444, pp. 341, 342.)

1. Ans. 0. (Art. 442.)

2. $-2-3-4 = -9$; hence $-8x^2$ should be $-9x^2$.

$$2 \times 3 + 2 \times 4 + 3 \times 4 = 6 + 8 + 12 = 26.$$

$(-2)(-3)(-4) = -24$. The second term is, therefore, the only one that should be changed.

3. $-3-5+2-\frac{2}{3} = -6\frac{2}{3} = -\frac{20}{3}$, and as the equation is of the fourth degree, the second term must contain x^3.

$$(-3)(-5)(2)(-\tfrac{2}{3}) = -20, \text{ the last term.}$$

4. $(-2)(-3)+(-2)4+(-3)4 = 6-8-12 = -14.$

$$2 \times 3 \times (-4) = -24.$$

5. $-1-2-3 = -6$; hence the second term is $-6x^2$.

$1 \times 2 + 1 \times 3 + 2 \times 3 = 2 + 3 + 6 = 11$; hence the third term is $11x$.

$(-1)(-2)(-3) = -6$, the last term.

6. As the absolute term is wanting, at least one root must be 0. (Art. 442.) As the term containing x is also wanting, two roots must be 0. Dividing by $x^2 - 0$, or x^2, we obtain the

Depressed equation, $x^2 + 8x + 10 = 0$
Completing the square, $x^2 + 8x + 16 = 6$
Evolving, $x + 4 = \pm\sqrt{6}$
Whence, $x = -4 \pm \sqrt{6}$

7. $-3-(-1-2)=0=$ coefficient of second term.
$-32 \div [(-1)(-2)] = -32 \div 2 = -16 =$ absolute term.

Hence the depressed equation is $x^2-16=0$, whose roots are 4 and -4. Ans. $x^2-16=0$.

IMAGINARY ROOTS.

(Art. 451, p. 344.)

1. Dividing x^3+x^2+4 by $x+2$, we obtain x^2-x+2.

Depressed equation, $x^2-x+2=0$
Completing square, $x^2-x+\frac{1}{4}=-\frac{7}{4}$
Whence, $x=\dfrac{1\pm\sqrt{-7}}{2}$

2. By Art. 448, one of the other roots must be $4-\sqrt{30}$. The product of two quantities with the signs changed is the same as the product of those quantities. The product of $4+\sqrt{30}$ and $4-\sqrt{30}$ is $16-30$, or -14. As there can be but three roots (Art. 436), the remaining root (Art. 441), must then be $-56 \div (-14) = 4$.

3. By Art. 448, one of the other roots must be $2+2\sqrt{2}$. The sum of the pair of known roots is 4, and their product is -4. The depressed equation may be obtained by either Art. 444 or Art. 439. If the latter method is used, it will be better to divide by $[x-(2+2\sqrt{2})][x-(2-2\sqrt{2})]$, or x^2-4x-4.

Depressed equation, $x^2-x-2=0$
Completing square, $x^2-x+\frac{1}{4}=\frac{9}{4}$
Evolving $x-\frac{1}{2}=\pm\frac{3}{2}$
Whence, $x=2$, and -1

4. By Art. 447, one of the other roots must be $-1-3\sqrt{-1}$;

and the product of the pair of imaginary roots is 10. As there can be but three roots (Art. 436), the remaining root (Art. 441), is $-50 \div 10 = -5$.

Ans. $-1 - 3\sqrt{-1}$, and -5.

DESCARTES' RULE.

(Art. 455, pp. 346, 347.)

1. It can have no negative root because it has no *permanence* of sign (Arts. 452, 433); and it can have no more than three positive roots because it has only three *variations* of sign.

2. Whatever sign is used for the second term, there will be one variation and two permanences; hence there must be one positive root and two negative ones.

3. Given $x^3 \pm 0x^2 - 10x + 3 = 0$. Whether the sign of the second term is taken as + or — there will in each case be two variations of sign and one permanence.

4. Given $x^3 - 7x^2 \pm 0x + 36 = 0$, in which there are two variations and one permanence.

Ans. Two positive; one negative.

5. Given $x^3 - x^2 \pm 0x - 4 = 0$. Taking the upper sign of $0x$ there is no permanence, and taking the lower sign there is but one variation; hence there is no negative root and but one positive root. The remaining two (Art. 436) must therefore be imaginary (Art. 453).

6. There are four variations and one permanence.

7. There are two permanences and no variation.

Ans. Both negative.

TRANSFORMATION OF EQUATIONS.

(Art. 457, pp. 347, 348.)

2. Ans. $x^4 + 2x^3 - x - 132 = 0$.

3. Ans. -5, -2, and 1.

(Art. 459, p. 349.)

All these equations may be transformed by a direct application of the rule found in Art. 458.

3. As the term containing x^2 is missing, -7 must be multiplied by 4^2, and -6 by 4^3.

4. Multiply $-\frac{3}{2}$ by $\frac{1}{3}$ and -27 by $(\frac{1}{3})^2$ for the coefficients of the transformed equation.

(Art. 460, pp. 349, 350.)

2. Assume the factor 2, that is, transform the equation into one whose roots are twice as great. As the second term is wanting, $\frac{3}{4} \times 2^2$ and $-\frac{7}{4} \times 2^3$ are the coefficients of the last two terms.

3. Assume the factor 2; then the coefficients of the transformed equation are -5×2, $\frac{35}{4} \times 2^2$, $-\frac{25}{4} \times 2^3$, and $\frac{3}{2} \times 2^4$.

4. Assume the factor bn; then the coefficients of the transformed equation are $-\frac{m}{n} \times bn$ and $\frac{a}{b} \times b^2 n^2$.

5. Assume the factor 30; then the coefficients of the transformed equation are $\frac{1}{30} \times 30$, $-\frac{1}{5} \times 30^2$, $-\frac{1}{30} \times 30^3$.

(Art. 461, pp. 350, 351.)

1. The first form is $-6y^3 + 11y^2 - 6y + 1 = 0$, and dividing by -6 we obtain $y^3 - \frac{11}{6}y^2 + y - \frac{1}{6} = 0$.

2. The first form is $2y^4 + y^3 - 3y^2 - y + 1 = 0$, and dividing by 2 we obtain $y^4 + \frac{1}{2}y^3 - \frac{3}{2}y^2 - \frac{1}{2}y + \frac{1}{2} = 0$.

3. The first form is $-\frac{1}{49}y^3 + \frac{6}{7}y^2 - 9y + 1 = 0$, and dividing by $-\frac{1}{49}$, or multiplying by -49, we obtain $y^3 - 42y^2 + 441y - 49 = 0$.

4. The first form is $9y^3 \pm 0y^2 - 4y + 1 = 0$, and dividing by 9 we obtain $y^3 - \frac{4}{9}y + \frac{1}{9} = 0$.

(Art. **465**, pp. 354–356.)

(4.)

$$\begin{array}{rrrrl}
1 & -9 & +26 & -24 & \big|\, \underline{+1} \\
 & \underline{+1} & \underline{-8} & \underline{+18} & \\
 & -8 & +18 & -6 = v' & \\
 & \underline{+1} & \underline{-7} & & \\
 & -7 & +11 = u' & & \\
 & \underline{+1} & & & \\
 & -6 = t' & & &
\end{array}$$

Hence, according to Art. 463, the required equation is $y^3 - 6y^2 + 11y - 6 = 0$.

(5.)

$$\begin{array}{rrrrl}
1 & -13 & +56 & -80 & \big|\, \underline{+3} \\
 & \underline{+3} & \underline{-30} & \underline{+78} & \\
 & -10 & +26 & -2 = v' & \\
 & \underline{+3} & \underline{-21} & & \\
 & -7 & +5 = u' & & \\
 & \underline{+3} & & & \\
 & -4 = t' & & &
\end{array}$$

Hence, according to Art. 463, the required equation is $y^3 - 4y^2 + 5y - 2 = 0$.

(6.)

$$\begin{array}{llllll}
1 & \pm 0 & -5 & \pm 0 & +4 & \underline{|-2} \\
 & \underline{-2} & \underline{+4} & \underline{+2} & \underline{-4} & \\
 & -2 & -1 & +2 & 0 = v' & \\
 & \underline{-2} & \underline{+8} & \underline{-14} & & \\
 & -4 & +7 & -12 = u' & & \\
 & \underline{-2} & \underline{+12} & & & \\
 & -6 & +19 = t' & & & \\
 & \underline{-2} & & & & \\
 & -8 = s' & & & &
\end{array}$$

Hence the required equation is $y^4 - 8y^3 + 19y^2 - 12y \pm 0 = 0$.

All the above examples can also be performed according to the methods indicated in the first and second solutions of the 1st example.

(ART. **466**, pp. 356–358.)

2. We substitute $y + \frac{p}{2}$ for x, as follows:

$$\begin{array}{llll}
1 & -p & +q & \underline{|+\frac{1}{2}p} \\
 & \underline{+\frac{1}{2}p} & \underline{-\frac{1}{4}p^2} & \\
 & -\frac{1}{2}p & -\frac{1}{4}p^2 + q & \\
 & \underline{+\frac{1}{2}p} & & \\
 & \pm 0 & &
\end{array}$$

Hence the required equation is $y^2 \pm 0\,y - \frac{1}{4}p^2 + q = 0$.

3. We substitute $y - \frac{6}{3}$, or $y - 2$, for x, as follows:

1	$+6$	-3	$+4$	-2
	-2	-8	$+22$	
	$+4$	-11	$+26$	
	-2	-4		
	$+2$	-15		
	-2			
	± 0			

Hence the required equation is $y^3 \pm 0y^2 - 15y + 26 = 0$.

4. We substitute $y + \frac{8}{4}$, or $y + 2$, for x, as follows:

1	-8	$+19$	-12	± 0	$+2$
	$+2$	-12	$+14$	$+4$	
	-6	$+7$	$+2$	$+4$	
	$+2$	-8	-2		
	-4	-1	± 0		
	$+2$	-4			
	-2	-5			
	$+2$				
	± 0				

Hence the required equation is $y^4 \pm 0y^3 - 5y^2 \pm 0y + 4 = 0$.

5. We substitute $y + 2$ for x, as follows:

1	-8	$+23$	-30	$+15$	$+2$
	$+2$	-12	$+22$	-16	
	-6	$+11$	-8	-1	
	$+2$	-8	$+6$		
	-4	$+3$	-2		
	$+2$	-4			
	-2	-1			
	$+2$				
	± 0				

Hence the required equation is $y^4 \pm 0y^3 - y^2 - 2y - 1 = 0$.

6. We substitute $y-\frac{1}{3}$ for x, as follows:

$$\begin{array}{rrrrl}
1 & +1 & \pm 0 & +\ \ 4 & \vert\ \underline{-\frac{1}{3}} \\
 & -\frac{1}{3} & -\frac{2}{9} & +\frac{2}{27} & \\
\cline{2-4}
 & +\frac{2}{3} & -\frac{2}{9} & +\frac{110}{27} & \\
 & -\frac{1}{3} & -\frac{1}{9} & & \\
\cline{2-3}
 & +\frac{1}{3} & -\frac{1}{3} & & \\
 & -\frac{1}{3} & & & \\
\cline{2-2}
 & \pm 0 & & &
\end{array}$$

Hence the required equation is $y^3 \pm 0y^2 - \frac{1}{3}y + \frac{110}{27} = 0$.

DERIVED POLYNOMIALS.

(Art. 468, pp. 358, 359.)

2. Ans. First, $8x^3-9x^2-8x+5$; second, $24x^2-18x-8$; third, $48x-18$; fourth, 48.

(Art. 469, pp. 359, 360.)

2. Here $r=4$ (Art. 462). The successive derivatives of the first member of the given equation are $4x^3-24x^2+48x-32$, $12x^2-48x+48$, $24x-48$, and 24. Hence

$$R = r^4 - 8r^3 + 24r^2 - 32r - 65$$
$$= 256 - 512 + 384 - 128 - 65 = -65$$
$$R_1 = 4r^3 - 24r^2 + 48r - 32 = 256 - 384 + 192 - 32 = 32$$
$$R_2 = 12r^2 - 48r + 48 = 192 - 192 + 48 = 48$$
$$R_3 = 24r - 48 = 96 - 48 = 48$$
$$R_4 = 24$$

Substituting these values in formula (5), we obtain

$$-65 + 32y + \frac{48}{2}y^2 + \frac{48}{6}y^3 + \frac{24}{24}y^4 = 0, \text{ or}$$
$$y^4 + 8y^3 + 24y^2 + 32y - 65 = 0$$

3. We must here substitute $y+\frac{7}{3}$ for x (Art. 466), that is, $r=\frac{7}{3}$ (Art. 462). The successive derivatives of the first member of the given equation are $3x^2-14x$, $6x-14$, and 6. Hence

$$\begin{aligned}
R &= r^3-7r^2+36=\tfrac{343}{27}-\tfrac{343}{9}+36=\tfrac{286}{27}\\
R_1 &= 3r^2-14r=\tfrac{49}{3}-\tfrac{98}{3}=-\tfrac{49}{3}\\
R_2 &= 6r-14=14-14=0\\
R_3 &= 6
\end{aligned}$$

Substituting these values in formula (5), we obtain

$$\tfrac{286}{27}-\tfrac{49}{3}y\pm 0y^2+\tfrac{6}{6}y^3=0, \text{ or } y^3-\tfrac{49}{3}y+\tfrac{286}{27}=0.$$

EQUAL ROOTS.

(ARTS. 471, 472, pp. 361, 362.)

1. The greatest common divisor is thus obtained:

$$\begin{array}{rl}
4x^3-42x^2+122x-84)\, & 4x^4-56x^3+244x^2-336x+144\,(x\\
 & \underline{4x^4-42x^3+122x^2-84x}\\
 & -14x^3+122x^2-252x+144\\
 & \underline{2}\\
 & -28x^3+244x^2-504x+288\,(-7\\
 & \underline{-28x^3+294x^2-854x+588}\\
 & -50\,)-50x^2+350x-300\\
 & x^2-7x+6
\end{array}$$

$$\begin{array}{rl}
x^2-7x+6\,)\, & 4x^3-42x^2+122x-84\,(\,4x-14\\
 & \underline{4x^3-28x^2+24x}\\
 & -14x^2+98x-84\\
 & \underline{-14x^2+98x-84}
\end{array}$$

2. The first derivative of $x^3-8x^2+13x-6$ is $3x^2-16x+13$.

$$\begin{array}{rll}
 & x^3-8x^2+13x-6 & \\
 & \underline{\phantom{x^3-8x^2+13x-{}}3} & \\
3x^2-16x+13\,) & 3x^3-24x^2+39x-18 & (\,x \\
 & \underline{3x^3-16x^2+13x} & \\
 & -8x^2+26x-18 & \\
 & \underline{3} & \\
 & -24x^2+78x-54 & (\,-8 \\
 & \underline{-24x^2+128x-104} & \\
 & -50\,)\,\underline{-50x+50} & \\
 & x-1\,)\;3x^2-16x+13 & (\,3x-13 \\
 & \underline{3x^2-3x} & \\
 & -13x+13 & \\
 & \underline{-13x+13} &
\end{array}$$

As $x-1$ is the greatest common divisor, then $x-1=0$, or $x=1$, and two of the roots of the original equation are 1. Dividing $x^3-8x^2+13x-6$ by $(x-1)^2$, or x^2-2x+1, we obtain $x-6$; hence, $x-6=0$, and $x=6$. Therefore the three roots are 1, 1, and 6.

3. The first derivative of $x^3-7x^2+16x-12$ is $3x^2-14x+16$.

$$\begin{array}{rll}
 & x^3-7x^2+16x-12 & \\
 & \underline{3} & \\
3x^2-14x+16\,) & 3x^3-21x^2+48x-36 & (\,x \\
 & \underline{3x^3-14x^2+16x} & \\
 & -7x^2+32x-36 & \\
 & \underline{3} & \\
 & -21x^2+96x-108 & (\,-7 \\
 & \underline{-21x^2+98x-112} & \\
 & -2\,)\,\underline{-2x+4} & \\
 & x-2\,)\;3x^2-14x+16 & (\,3x-8 \\
 & \underline{3x^2-6x} & \\
 & -8x+16 & \\
 & \underline{-8x+16} &
\end{array}$$

As $x - 2$ is the greatest common divisor, then $x - 2 = 0$, or $x = 2$, and two of the roots of the original equation are 2. Dividing $x^3 - 7x^2 + 16x - 12$ by $(x - 2)^2$, or $x^2 - 4x + 4$, we obtain $x - 3$; hence $x - 3 = 0$, and $x = 3$. Therefore the three roots are 2, 2, and 3.

4. The first derivative of $x^4 - x^3 - 18x^2 + 52x - 40$ is $4x^3 - 3x^2 - 36x + 52$.

$$\begin{array}{rrl}
 & x^4 - x^3 - 18x^2 + 52x - 40 & \\
 & 4 & \\ \cline{2-2}
4x^3 - 3x^2 - 36x + 52\,) & 4x^4 - 4x^3 - 72x^2 + 208x - 160 & (\,x \\
 & 4x^4 - 3x^3 - 36x^2 + 52x\phantom{{}-160} & \\ \cline{2-2}
 & -x^3 - 36x^2 + 156x - 160 & \\
 & 4 & \\ \cline{2-2}
 & -4x^3 - 144x^2 + 624x - 640 & (\,-1 \\
 & -4x^3 + 3x^2 + 36x - 52 & \\ \cline{2-2}
-147\,) & -147x^2 + 588x - 588 & \\ \cline{2-2}
 & x^2 - 4x + 4 &
\end{array}$$

$$\begin{array}{rrl}
x^2 - 4x + 4\,) & 4x^3 - 3x^2 - 36x + 52 & (\,4x + 13 \\
 & 4x^3 - 16x^2 + 16x\phantom{{}+52} & \\ \cline{2-2}
 & 13x^2 - 52x + 52 & \\
 & 13x^2 - 52x + 52 & \\ \cline{2-2}
\end{array}$$

As $x^2 - 4x + 4$ is the greatest common divisor, then $x^2 - 4x + 4 = 0$, or $(x - 2)^2 = 0$, whence $x = 2$ and 2, and three of the roots of the original equation must be 2. Dividing $x^4 - x^3 - 18x^2 + 52x - 40$ by $(x - 2)^3$, or $x^3 - 6x^2 + 12x - 8$, we obtain $x + 5$; hence $x + 5 = 0$, and $x = -5$. Therefore the four roots are 2, 2, 2, and -5.

5. The first derivative of $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108$ is $5x^4 - 52x^3 + 201x^2 - 342x + 216$.

$$\begin{array}{l|l}
 & x^5-13x^4+67x^3-171x^2+216x-108 \\
 & \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5} \\
5x^4-52x^3+201x^2-342x & 5x^5-65x^4+335x^3-855x^2+1080x-540\,(x \\
+216 & \underline{5x^5-52x^4+201x^3-342x^2+216x} \\
 & -13x^4+134x^3-513x^2+864x-540 \\
 & \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5} \\
 & -65x^4+670x^3-2565x^2+4320x-2700\,(-13 \\
 & \underline{-65x^4+676x^3-2613x^2+4446x-2808} \\
 & -6)\,\underline{-6x^3+48x^2-126x+108} \\
 & \qquad x^3-8x^2+21x-18
\end{array}$$

$$\begin{array}{l}
x^3-8x^2+21x-18\,)\,5x^4-52x^3+201x^2-342x+216\,(\,5x-12 \\
\qquad\qquad\qquad\qquad\;\; \underline{5x^4-40x^3+105x^2-90x} \\
\qquad\qquad\qquad\qquad\qquad -12x^3+96x^2-252x+216 \\
\qquad\qquad\qquad\qquad\qquad \underline{-12x^3+96x^2-252x+216}
\end{array}$$

As $x^3-8x^2+21x-18$, the greatest common divisor, is of the third degree, it must be compared with its own derivative, $3x^2-16x+21$.

$$\begin{array}{l}
\qquad\qquad\qquad\quad x^3-8x^2+21x-18 \\
\qquad\qquad\qquad\quad \underline{\qquad\qquad\qquad\quad 3} \\
3x^2-16x+21\,)\,3x^3-24x^2+63x-54\,(\,x \\
\qquad\qquad\qquad\quad \underline{3x^3-16x^2+21x} \\
\qquad\qquad\qquad\qquad -8x^2+42x-54 \\
\qquad\qquad\qquad\qquad \underline{\qquad\qquad\qquad 3} \\
\qquad\qquad\qquad\qquad -24x^2+126x-162\,(-8 \\
\qquad\qquad\qquad\qquad \underline{-24x^2+128x-168} \\
\qquad\qquad\qquad\qquad -2)\,\underline{-2x+6} \\
\qquad\qquad\qquad\qquad\qquad\qquad x-3\,)\,3x^2-16x+21\,(\,3x-7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x^2-9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -7x+21 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-7x+21}
\end{array}$$

Then $x-3=0$, or $x=3$; hence two of the roots of $x^3-8x^2+21x-18=0$ are 3, and three of the roots of the original equation must be 3. Dividing $x^3-8x^2+21x-18$ by $(x-3)^2$, or x^2-6x+9, we obtain $x-2$; hence $x-2=0$, and $x=2$. One root of the equation $x^3-8x^2+21x-18=0$ is then 2, and the four roots of the original equation are 2, 2, 3, 3, and 3.

6.
$$
\begin{array}{l}
x^4+3x^3-13x^2-27x+36 \\
\underline{x^4-3x^3-13x^2+27x+36} \\
6x\,)\,\underline{6x^3 \qquad -54x} \\
\qquad x^2 \qquad -9\,)\,x^4+3x^3-13x^2-27x+36\,(\,x^2+3x-4 \\
\qquad\qquad\qquad\qquad \underline{x^4 \qquad -9x^2} \\
\qquad\qquad\qquad\qquad\quad 3x^3-4x^2-27x \\
\qquad\qquad\qquad\qquad\quad \underline{3x^3 \qquad\ -27x} \\
\qquad\qquad\qquad\qquad\qquad -4x^2 \qquad +36 \\
\qquad\qquad\qquad\qquad\qquad \underline{-4x^2 \qquad +36}
\end{array}
$$

It is evident that the original equation has thus been separated into $x^2-9=0$ and $x^2+3x-4=0$.

The solution of the former gives $x=\pm 3$, and of the latter gives $x=1$ and -4.

SIGNS OF POLYNOMIAL FUNCTIONS.

(Art. 474, pp. 363, 364.)

2. Here, $x=1$ gives $X=7$, and $x=2$ gives $X=-1$; hence one root lies between 1 and 2, that is, its integral part is 1.

Again, $x=5$ gives $X=-1$, and $x=6$ gives $X=27$; hence one root lies between 5 and 6.

By Descartes' Rule (Art. 452) and Art. 450, it is evident that the remaining root is negative. $x=0$ gives $X=9$, and $x=-1$ gives $X=-1$; hence one root lies between 0 and -1.

Ans. One root lies between 1 and 2, one between 5 and 6, and one between 0 and -1.

STURM'S THEOREM.

(ART. 481, pp. 368–371.)

2. The first derivative of $x^4-3x^3+3x^2-3x+\frac{5}{2}$ is $4x^3-9x^2+6x-3$.

$$\begin{array}{rl}
 & x^4-3x^3+3x^2-3x+\tfrac{5}{2} \\
 & 4 \\ \cline{2-2}
4x^3-9x^2+6x-3\,) & 4x^4-12x^3+12x^2-12x+10\ (x \\
 & 4x^4-9x^3+6x^2-3x \\ \cline{2-2}
 & -3x^3+6x^2-9x+10 \\
 & 4 \\ \cline{2-2}
 & -12x^3+24x^2-36x+40\ (-3 \\
 & -12x^3+27x^2-18x+9 \\ \cline{2-2}
 & -3x^2-18x+31
\end{array}$$

Therefore $X_2 = 3x^2+18x-31$

$$\begin{array}{rl}
 & 4x^3-9x^2+6x-3 \\
 & 3 \\ \cline{2-2}
3x^2+18x-31\,) & 12x^3-27x^2+18x-9\ (4x-33 \\
 & 12x^3+72x^2-124x \\ \cline{2-2}
 & -99x^2+142x-9 \\
 & -99x^2-594x+1023 \\ \cline{2-2}
 & 8\,)\ 736x-1032 \\ \cline{2-2}
 & 92x-129\ \therefore\ X_3=-92x+129
\end{array}$$

$$\begin{array}{rl}
 & 3x^2+18x-31 \\
 & 92 \\ \cline{2-2}
-92x+129\,) & 276x^2+1656x-2852\ (-3x \\
 & 276x^2-387x \\ \cline{2-2}
 & 2043x-2852 \\
 & 92 \\ \cline{2-2}
 & 187956x-262384\ (-2043 \\
 & 187956x-263547 \\ \cline{2-2}
 & 1163\ \therefore\ X_4=-1163
\end{array}$$

3. The first derivative of $x^3 - x^2 - 2x + 1$ is $3x^2 - 2x - 2$.

$$\begin{array}{r r}
 & x^3 - x^2 - 2x + 1 \\
 & 3 \\ \cline{2-2}
3x^2 - 2x - 2\,) & 3x^3 - 3x^2 - 6x + 3\,(\,x \\
 & 3x^3 - 2x^2 - 2x \\ \cline{2-2}
 & -x^2 - 4x + 3 \\
 & 3 \\ \cline{2-2}
 & -3x^2 - 12x + 9\,(\,-1 \\
 & -3x^2 + 2x + 2 \\ \cline{2-2}
7\,) & -14x + 7 \\ \cline{2-2}
 & -2x + 1
\end{array}$$

$$\begin{array}{r r}
 & 3x^2 - 2x - 2 \\
 & 2 \\ \cline{2-2}
2x - 1\,) & 6x^2 - 4x - 4\,(\,3x \\
 & 6x^2 - 3x \\ \cline{2-2}
 & -x - 4 \\
 & 2 \\ \cline{2-2}
 & -2x - 8\,(\,-1 \\
 & -2x + 1 \\ \cline{2-2}
 & -9
\end{array}$$

Hence $X = x^3 - x^2 - 2x + 1$; $X_2 = 2x - 1$;

$X_1 = 3x^2 - 2x - 2$; $X_3 = +9$.

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -2$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -1$,	$+$	$+$	$-$	$+$,	2 variations.
$x = 0$,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1$,	$-$	$-$	$+$	$+$,	1 variation.
$x = 2$,	$+$	$+$	$+$	$+$,	no variation.
$x = +\infty$,	$+$	$+$	$+$	$+$,	no variation.

By the substitution of $-\infty$ and $+\infty$ for x, we obtain a difference of three variations; hence the three roots are all real. Substituting 0 for x we find that one variation is lost between $-\infty$ and 0, and two between 0 and $+\infty$; hence one root is negative and two positive. By substituting other numbers for x we find that one variation is lost between -2 and -1, one between 0 and 1, and one between 1 and 2.

4. The first derivative of x^3-7x+7 is $3x^2-7$.

$$\begin{array}{r}
x^3-7x+7\\
3\\
\hline
3x^2-7\,)\,3x^3-21x+21\,(x\\
3x^3-7x\\
\hline
7\,)\,\underline{-14x+21}\\
-2x+3
\end{array}
\qquad\qquad
\begin{array}{r}
3x^2-7\\
2\\
\hline
2x-3\,)\,6x^2-14\,(3x\\
6x^2-9x\\
\hline
9x-14\\
2\\
\hline
18x-28\,(9\\
18x-27\\
\hline
-1
\end{array}$$

Hence, $X = x^3-7x+7$; $X_2 = 2x-3$;

$X_1 = 3x^2-7$; $X_3 = +1$.

	X	X_1	X_2	X_3	
$x=-\infty$,	$-$	$+$	$-$	$+$,	3 variations.
$x=-4$,	$-$	$+$	$-$	$+$,	3 variations.
$x=-3$,	$+$	$+$	$-$	$+$,	2 variations.
$x=0$,	$+$	$-$	$-$	$+$,	2 variations.
$x=1$,	$+$	$-$	$-$	$+$,	2 variations.
$x=2$,	$+$	$+$	$+$	$+$,	no variation.
$x=+\infty$,	$+$	$+$	$+$	$+$,	no variation.

By substituting $-\infty$ and $+\infty$ we learn that all three of the roots are real; and by substituting other values of x we learn that one variation is lost between -4 and -3 and the two others between 1 and 2.

5. The first derivative of $x^4-2x^3-5x^2+10x-3$ is $4x^3-6x^2-10x+10$, and casting out the factor 2 we have $2x^3-3x^2-5x+5$.

$$
\begin{array}{rrl}
 & x^4-2x^3-5x^2+10x-3 & \\
 & 2 & \\ \cline{2-2}
2x^3-3x^2-5x+5\,) & 2x^4-4x^3-10x^2+20x-6 & (\,x \\
 & 2x^4-3x^3-5x^2+5x & \\ \cline{2-2}
 & -x^3-5x^2+15x-6 & \\
 & 2 & \\ \cline{2-2}
 & -2x^3-10x^2+30x-12 & (\,-1 \\
 & -2x^3+3x^2+5x-5 & \\ \cline{2-2}
 & -13x^2+25x-7 &
\end{array}
$$

$$
\begin{array}{rrl}
 & 2x^3-3x^2-5x+5 & \\
 & 13 & \\ \cline{2-2}
13x^2-25x+7\,) & 26x^3-39x^2-65x+65 & (\,2x \\
 & 26x^3-50x^2+14x & \\ \cline{2-2}
 & 11x^2-79x+65 & \\
 & 13 & \\ \cline{2-2}
 & 143x^2-1027x+845 & (\,11 \\
 & 143x^2-275x+77 & \\ \cline{2-2}
 & 16\,)\,-752x+768 & \\ \cline{2-2}
 & -47x+48 &
\end{array}
$$

$$
\begin{array}{rrl}
 & 13x^2-25x+7 & \\
 & 47 & \\ \cline{2-2}
47x-48\,) & 611x^2-1175x+329 & (\,13x-\frac{551}{47} \\
 & 611x^2-624x & \\ \cline{2-2}
 & -551x+329 & \\
 & -551x+ & \\ \cline{2-2}
 & - &
\end{array}
$$

Hence, $X = x^4 - 2x^3 - 5x^2 + 10x - 3$; $X_3 = 47x - 48$;
$X_1 = 2x^3 - 3x^2 - 5x + 5$; $X_4 = +$.
$X_2 = 13x^2 - 25x + 7$;

	X	X_1	X_2	X_3	X_4	
$x = -\infty$,	$+$	$-$	$+$	$-$	$+$,	4 variations.
$x = -3$,	$+$	$-$	$+$	$-$	$+$,	4 variations.
$x = -2$,	$-$	$-$	$+$	$-$	$+$,	3 variations.
$x = 0$,	$-$	$+$	$+$	$-$	$+$,	3 variations.
$x = 1$,	$+$	$-$	$-$	$-$	$+$,	2 variations.
$x = 2$,	$-$	$-$	$+$	$+$	$+$,	1 variation.
$x = 3$,	$+$	$+$	$+$	$+$	$+$,	no variation.
$x = +\infty$,	$+$	$+$	$+$	$+$	$+$,	no variation.

By substituting $-\infty$ and $+\infty$ we learn that all four of the roots are real; and by substituting other values of x we learn that the variations are lost between -3 and -2, 0 and 1, 1 and 2, 2 and 3.

6. The first derivative of $x^3 - 2x - 5$ is $3x^2 - 2$.

$$\begin{array}{r|l}
 & x^3 - 2x - 5 \\
 & \qquad\qquad\; 3 \\
\hline
3x^2 - 2 \;) & 3x^3 - 6x - 15 \;(x \\
 & 3x^3 - 2x \\
\hline
 & \quad -4x - 15
\end{array}
\qquad
\begin{array}{r|l}
 & 3x^2 - 2 \\
 & \qquad\quad\; 4 \\
\hline
4x + 15 \;) & 12x^2 - 8 \;(3x \\
 & 12x^2 + 45x \\
\hline
 & \quad -45x - 8 \\
 & \qquad\qquad\;\; 4 \\
\hline
 & -180x - 32 \;(-45 \\
 & -180x - 675 \\
\hline
 & \qquad\quad +643
\end{array}$$

Hence, $X = x^3 - 2x - 5$; $X_2 = 4x + 15$;
$X_1 = 3x^2 - 2$; $X_3 = -643$.

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$-$,	2 variations.
$x = 0$,	$-$	$-$	$+$	$-$,	2 variations.
$x = 2$,	$-$	$+$	$+$	$-$,	2 variations.
$x = 3$,	$+$	$+$	$+$	$-$,	1 variation.
$x = +\infty$,	$+$	$+$	$+$	$-$,	1 variation.

As only one variation is lost between $-\infty$ and $+\infty$, there can only one real root, and this is found, by substituting other values for x, to lie between 2 and 3.

7. The first derivative of $2x^4 - 3x^3 + 17x^2 - 3x + 15$ is $8x^3 - 9x^2 + 34x - 3$.

$$\begin{array}{rr}
 & 2x^4 - 3x^3 + 17x^2 - 3x + 15 \\
 & 4 \\ \cline{2-2}
8x^3 - 9x^2 + 34x - 3\,) & 8x^4 - 12x^3 + 68x^2 - 12x + 60\,(\,x \\
 & 8x^4 - 9x^3 + 34x^2 - 3x \\ \cline{2-2}
 & -3x^3 + 34x^2 - 9x + 60 \\
 & 8 \\ \cline{2-2}
 & -24x^3 + 272x^2 - 72x + 480\,(\,-3 \\
 & -24x^3 + 27x^2 - 102x + 9 \\ \cline{2-2}
 & 245x^2 + 30x + 471 \\
 & 8x^3 - 9x^2 + 34x - 3 \\
 & 245 \\ \cline{2-2}
-245x^2 - 30x - 471\,) & 1960x^3 - 2205x^2 + 8330x - 735\,(\,-8x \\
 & 1960x^3 + 240x^2 + 3768x \\ \cline{2-2}
 & -2445x^2 + 4562x - 735 \\
 & 49 \\ \cline{2-2}
 & -119805x^2 + 223538x - 36015\,(\,489 \\
 & -119805x^2 - 14670x - 230319 \\ \cline{2-2}
 & 128\,)\,238208x + 194304 \\ \cline{2-2}
 & 1861x + 1518
\end{array}$$

$$\begin{array}{rl}
 & -245x^2 \quad -30x \quad -471 \\
 & \hfill 1861 \\
\hline
-1861x-1518\,) & -455945x^2 - 55830x - 876531\;(\,245x - \\
 & -455945x^2 - 371910x \\
\hline
 & \hfill 316080x - 876531 \\
 & \hfill + \\
\hline
\end{array}$$

It is evident that the remainder will be $-$, and therefore X_4 must be $+$.

Hence $X=2x^4-3x^3+17x^2-3x+15$; $X_3=-1861x-1518$;

$X_1=8x^3-9x^2+34x-3$; $X_4=+$.

$X_2=-245x^2-30x-471$;

	X	X_1	X_2	X_3	X_4	
$x=-\infty$,	$+$	$-$	$-$	$+$	$+$,	2 variations.
$x=+\infty$,	$+$	$+$	$-$	$-$	$+$,	2 variations.

As no variation is lost between $-\infty$ and $+\infty$, there can be no real root.

8. The first derivative of $x^4-4x^3-3x+27$ is $4x^3-12x^2-3$.

$$\begin{array}{rl}
 & x^4 - 4x^3 - 3x + 27 \\
 & \hfill 4 \\
\hline
4x^3-12x^2-3\,) & 4x^4 - 16x^3 - 12x + 108\;(\,x-1 \\
 & 4x^4 - 12x^3 - 3x \\
\hline
 & -4x^3 - 9x + 108 \\
 & -4x^3 + 12x^2 + 3 \\
\hline
3\,) & -12x^2 - 9x + 105 \\
\hline
 & -4x^2 - 3x + 35
\end{array}$$

$$\begin{array}{r}
4x^2+3x-35\,)\,4x^3-12x^2\quad-3\,(\,x\\
4x^3+\ 3x^2\ -35x\\
\hline
-15x^2\ +35x\ -\ 3\\
4\\
\hline
-60x^2+140x\ -12\,(-15\\
-60x^2-\ \ 45x+525\\
\hline
185x-537
\end{array}$$

$$\begin{array}{r}
4x^2\ +\ 3x\ \ -35\phantom{\,(-4x-\frac{2703}{185}}\\
185\phantom{\,(-4x-\frac{2703}{185}}\\
\hline
-185x+537\,)\,740x^2+\ 555x-6475\,(-4x-\tfrac{2703}{185}\\
740x^2-2148x\phantom{\,(-4x-\frac{2703}{185}}\\
\hline
2703x-6475\phantom{\,(-4x-\frac{2703}{185}}\\
2703x-7846+\phantom{\,(-4x-\frac{2703}{185}}\\
\hline
+\phantom{\,(-4x-\frac{2703}{185}}
\end{array}$$

Hence $X = x^4 - 4x^3 - 3x + 27$; $X_3 = -185x + 537$;

$X_1 = 4x^3 - 12x^2 - 3$; $X_4 = -$.

$X_2 = 4x^2 + 3x - 35$;

	X	X_1	X_2	X_3	X_4	
$x = -\infty$,	+	−	+	+	−,	3 variations.
$x = 0$,	+	−	−	+	−,	3 variations.
$x = 1$,	+	−	−	+	−,	3 variations.
$x = 2$,	+	−	−	+	−,	3 variations.
$x = 3$,	−	−	+	−	−,	2 variations.
$x = 4$,	+	+	+	−	−,	1 variation.
$x = +\infty$,	+	+	+	−	−,	1 variation.

As only two variations are lost between $-\infty$ and $+\infty$, there can be only two real roots. By substituting other values for x, these variations are found to be lost between 2 and 3, and 3 and 4.

SOLUTION OF HIGHER NUMERICAL EQUATIONS.

COMMENSURABLE ROOTS.

(ART. 485, pp. 373–376.)

3. It is evident, by Descartes' Rule (Art. 452), that there can be no positive root. The only negative integral divisors of 6, the absolute term, are -1, -2, -3, and -6.

Divisors,	-1,	-2,	-3,	-6
1st Quotients,	-6,	-3,	-2,	-1
Quotients $+11$,	$+5$,	$+8$,	$+9$,	$+10$
2d Quotients,	-5,	-4,	-3	
Quotients $+6$,	$+1$,	$+2$,	$+3$	
3d Quotients,	-1,	-1,	-1	
Quotients $+1$,	0,	0,	0	

Hence the divisors -1, -2, and -3 are the three roots of the equation.

The three roots may also be obtained by Art. 484, as in the solution of Example 1.

4. As the sum of the coefficients is not 0, 1 cannot be a root. The substitution of -1 also proves that it is not a root. The remaining divisors of -12, the absolute term, are ± 2, ± 3, ± 6, and ± 12.

Divisors,	+12,	+6,	+3,	+2,	−2,	−3,	−6,	−12
1st Quotients,	−1,	−2,	−4,	−6,	+6,	+4,	+2,	+1
Quo. −4,	−5,	−6,	−8,	−10,	+2,	0,	−2,	−3
2d Quotients,		−1,		−5,	−1,	0		
Quotients +3,		+2,		−2,	+2,	3		
3d Quotients,				−1,	−1,	−1		
Quotients +1,				0,	0,	0		

Hence the divisors 2, −2, and −3 are the three roots of the equation.

5. It is evident, by Descartes' Rule, that the equation can have no negative root.

Pos. Divisors,	+15,	+5,	+3,	+1
1st Quotients,	+1,	+3,	+5,	+15
Quotients −20,	−19,	−17,	−15,	−5
2d Quotient,			−5,	−5
Quotient +23,			+18,	+18
3d Quotient,			+6,	+18
Quotient −9,			−3,	+9
4th Quotient,			−1,	+9

Hence the divisor 3 is the only commensurable root of the equation.

6. As the sum of the coefficients is not 0, 1 can not be a root of the equation. We next make trial of 2, according to Art. 484. Dividing the first member by $x-2$,

$$\begin{array}{rrrr|l} 1 & -3 & +1 & +2 & +2 \\ & +2 & -2 & -2 & \\ \hline 1 & -1 & -1, & 0 & \end{array}$$

we obtain x^2-x-1, and no remainder; hence 2 is a root of the equation, and $x^2-x-1=0$ contains the other two roots.

Depressed equation, $x^2 - x - 1 = 0$

Completing square, $x^2 - x + \frac{1}{4} = \frac{5}{4}$

Whence, $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} = 1.618$, or -0.618

7. It is easily proved by substitution that $+1$ and -1 are not roots.

	+32,	+16,	+8,	+4,	+ 2,	− 2,	− 4,	− 8,	−16,	−32
	+ 1,	+ 2,	+4,	+8,	+16,	−16,	− 8,	− 4,	− 2,	− 1
	− 7,	− 6,	−4,	0,	+ 8,	−24,	−16,	−12,	−10,	− 9
2d Quotients,				0,	+ 4,	+12,	+ 4			
Quotients + 0,				0,	+ 4,	+12,	+ 4			
3d Quotients,				0,	+ 2,	− 6,	− 1			
Quotients − 4,				−4,	− 2,	−10,	− 5			
4th Quotients,				−1,	− 1,	+ 5				
Quotients + 1,				0,	0,	6				

Hence the divisors 4 and 2 are the only commensurable roots. Dividing $x^4 - 4x^3 - 8x + 32$ by $x - 4$ and by $x - 2$, or by $x^2 - 6x + 8$, we obtain $x^2 + 2x + 4$.

Depressed equation, $x^2 + 2x + 4 = 0$

Completing square, $x^2 + 2x + 1 = -3$

Whence, $x = -1 \pm \sqrt{-3}$

8. It is easily proved that $+1$ and -1 are not roots.

Pos. Div.,	+36,	+18,	+12,	+9,	+6,	+4,	+ 3,	+ 2
1st Quo.,	+ 1,	+ 2,	+ 3,	+4,	+6,	+9,	+12,	+18
Quo. + 0,	+ 1,	+ 2,	+ 3,	+4,	+6,	+9,	+12,	+18
2d Quotients,					+1,		+ 4,	+ 9
Quotients − 7,					−6,		− 3,	+ 2
3d Quotients,					−1,		− 1,	+ 1
Quotients + 1,					0,		0,	2

Hence 6 and 3 are roots of the equation.

Neg. Div.	-2,	-3,	-4,	-6,	-9,	-12,	-18,	-36
1st Quotients,	-18,	-12,	-9,	-6,	-4,	-3,	-2,	-1
Quo. $+0$,	-18,	-12,	-9,	-6,	-4,	-3,	-2,	-1
2d Quotients,	$+9$,	$+4$,		$+1$				
Quo. -7,	$+2$,	-3,		-6				
3d Quotients,	-1,	$+1$,		$+1$				
Quo. $+1$,	0,	2,		$+2$				

Hence -2 is a root of the equation.

9. By Descartes' Rule it is evident that there can be no negative root.

Positive Divisors,	$+1$,	$+2$,	$+4$,	$+8$
1st Quotients,	-8,	-4,	-2,	-1
Quotients $+10$,	$+2$,	$+6$,	$+8$,	$+9$
2d Quotients,	$+2$,	$+3$,	$+2$	
Quotients -6,	-4,	-3,	-4	
3d Quotients,	-4,		-1	
Quotients $+1$,	-3,		0	

Hence 4 is the only commensurable root of the equation.

Dividing $x^3-6x^2+10x-8$ by $x-4$, we obtain x^2-2x+2.

Depressed equation, $x^2-2x+2=0$

Completing square, $x^2-2x+1=-1$

Whence, $x=1\pm\sqrt{-1}$

10. It is easily proved that 1 and -1 are not roots. The number of divisors of 180 is too great to use them all at once; hence we select the smaller ones.

Divisors,	+ 5,	+ 4,	+ 3,	+ 2,	— 2,	— 3,	— 4,	— 5
1st Quo.,	+36,	+45,	+60,	+90,	—90,	—60,	—45,	—36
Quo. —9,	+27,	+36,	+51,	+81,	—99,	—69,	—54,	—45
2d Quotients,		+ 9,	+17,			+23,		+ 9
Quo. —29,		—20,	—12,			— 6,		—20
3d Quotients,		— 5,	— 4,			+ 2,		+ 4
Quo. +1,		— 4,	— 3,			+ 3,		+ 5
4th Quotients,		— 1,	— 1,			— 1,		— 1
Quo. +1,		0,	0,			0,		0

Hence 4, 3, —3, and —5 are the four roots of the equation, and there is no necessity for trying the other divisors of 180.

11. It is evident, by Descartes' Rule, that there can be no negative root.

Positive Divisors,	+ 6,	+3,	+2,	+1
1st Quotients,	— 1,	—2,	—3,	—6
Quotients +11,	+10,	+9,	+8,	+5
2d Quotients,		+3,	+4,	+5
Quotients — 6,		—3,	—2,	—1
3d Quotients,		—1,	—1,	—1
Quotients +1,		0,	0,	0

Ans. 3, 2, and 1.

It is evident that the roots of the above equation are the same as those of the 3d, with their signs changed. (Art. 457.)

12. Putting $x = \frac{y}{2}$, we obtain (Art. 460),

$$y^3 - 8y^2 - 9y + 72 = 0,$$

whose coefficients are entire, that of the first term being 1, and whose roots are twice as great as those of the given equation.

After trying 1 and —1, we select the smaller divisors of 72

Divisors,	+8,	+ 6,	+ 4,	+ 3,	+ 2,	− 2,	− 3,	− 4
1st Quo.,	+9,	+12,	+18,	+24,	+36,	−36,	−24,	−18
Quo. −9,	0,	+ 3,	+ 9,	+15,	+27,	−45,	−33,	−27
2d Quo.,	0,			+ 5,			+11	
Quo. −8,	−8,			− 3,			+ 3	
3d Quo.,	−1,			− 1,			− 1	
Quo. +1,	0,			0,			0	

Hence 8, 3, and −3 are the roots of the transformed equation, and the roots of the given equation are one half as great, or 4, $\frac{3}{2}$, and $-\frac{3}{2}$.

13. Putting $x=\frac{y}{2}$, we obtain (Art. 460),

$$y^3-3y^2+32y-96=0.$$

Selecting the smaller divisors of −96,

Divisors,	+ 3,	+ 2,	+ 1,	− 1,	− 2,	− 3
1st Quotients,	−32,	−48,	−96,	+ 96,	+48,	+32
Quo. +32,	0,	−16,	−64,	+128,	+80,	+64
2d Quotients,	0,	− 8,	−64,	−128,	−40	
Quo. −3,	−3,	−11,	−67,	−131,	−43	
3d Quotients,	−1,		−67,	+131		
Quo. +1,	0,		−66,	+132		

Hence 3 is a root of the transformed equation. Dividing $y^3-3y^2+32y-96$ by $y-3$, we obtain y^2+32.

Depressed equation, $y^2+32=0$

Whence, $y=\pm\sqrt{-32}=\pm4\sqrt{-2}$

The three roots of the transformed equation are therefore 3 and $\pm4\sqrt{-2}$, and those of the original equation must be one half as great, or $\frac{3}{2}$ and $\pm2\sqrt{-2}$.

14. It is easily proved that 1 and −1 are not roots of the equation. The other divisors of −16 are ±2, ±4, ±8, ±16.

Divisors,	+16,	+8,	+4,	+2,	−2,	−4,	−8,	−16
1st Quotients,	− 1,	−2,	−4,	−8,	+8,	+4,	+2,	+ 1
Quo. +0,	− 1,	−2,	−4,	−8,	+8,	+4,	+2,	+ 1
2d Quotients,			−1,	−4,	−4,	−1		
Quotients +0,			−1,	−4,	−4,	−1		
3d Quotients,				−2,	+2			
Quotients −2,				−4,	0			
4th Quotients,				−2,	0			
Quotients +0,				−2,	0			
5th Quotients,				−1,	0			
Quotients +1,				0,	+1			

Hence 2 is the only commensurable root of the equation.

INCOMMENSURABLE ROOTS.

HORNER'S METHOD.

(Art. **490**, pp. 378–380.)

1. The first derivative of $x^3 - x^2 - 2x + 1$ is $3x^2 - 2x - 2$.

$$\begin{array}{rrrrrl}
 & x^3 & -\;x^2 & -\;2x & +1 & \\
 & & & & 3 & \\ \cline{2-5}
3x^2-2x-2\,) & 3x^3 & -3x^2 & -\;6x & +3 & (x \\
 & 3x^3 & -2x^2 & -\;2x & & \\ \cline{2-5}
 & - & x^2 & -\;4x & +3 & \\
 & & & & 3 & \\ \cline{2-5}
 & & -3x^2 & -12x & +9 & (-1 \\
 & & -3x^2 & +\;2x & +2 & \\ \cline{2-5}
 & & 7\,) & -14x & +7 & \\ \cline{3-5}
 & & - & 2x & +1 &
\end{array}$$

$$\begin{array}{rl}
 & 3x^2 - 2x - 2 \\
 & \qquad\qquad\quad 2 \\
\cline{2-2}
2x - 1\,) & 6x^2 - 4x - 4\,(\,3x \\
 & 6x^2 - 3x \\
\cline{2-2}
 & \qquad -\ x - 4 \\
 & \qquad\qquad\quad 2 \\
\cline{2-2}
 & \qquad -2x - 8\,(\,-1 \\
 & \qquad -2x + 1 \\
\cline{2-2}
 & \qquad\qquad -9
\end{array}$$

Hence $X = x^3 - x^2 - 2x + 1$; $\quad X_2 = 2x - 1$;
$X_1 = 3x^2 - 2x - 2$; $\quad X_3 = +9$.

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -2$,	$-$	$+$	$-$	$+$,	3 variations.
$x = -1$,	$+$	$+$	$-$	$+$,	2 variations.
$x = 0$,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1$,	$-$	$-$	$+$	$+$,	1 variation.
$x = 2$,	$+$	$+$	$+$	$+$,	no variation.
$x = +\infty$,	$+$	$+$	$+$	$+$,	no variation.

$$\begin{array}{ccccl}
1 & -1.0 & -2.00 & +1.000 & \lfloor .4 \\
 & .4 & -\ .24 & -.896 & \\
\cline{2-4}
 & -.6 & -2.24 & .104 & \\
 & .4 & -\ .08 & & \\
\cline{2-3}
 & -.2 & -2.32 & & \\
 & .4 & & & \\
\cline{2-2}
 & .2 & & & \\
\hline
1 & +.20 & -2.32 & +.104 & \lfloor .04 \\
 & .04 & .01 & -.092 & \\
\cline{2-4}
 & .24 & -2.31 & .012 & \\
 & .04 & .01 & & \\
\cline{2-3}
 & .28 & -2.30 & & \dfrac{.012}{2.3} = .005
\end{array}$$

Hence the second root is .445.

$$
\begin{array}{rrrrl}
1 & +1 & -2 & -1 & \lfloor \underline{1.} \\
 & \underline{1} & \underline{2} & \underline{0} & \\
 & 2 & 0 & -1 & \\
 & \underline{1} & \underline{3} & & \\
 & 3 & 3 & & \\
 & \underline{1} & & & \\
 & 4 & & & \\
\hline
1 & +4.00 & +3.00 & -1.000 & \lfloor \underline{.2} \\
 & \underline{.2} & \underline{.84} & \underline{.768} & \\
 & 4.2 & 3.84 & -.232 & \\
 & \underline{.2} & \underline{.88} & & \\
 & 4.4 & 4.72 & & \\
 & \underline{.2} & & & \\
 & 4.6 & & & \\
\hline
1 & +4.60 & +4.720 & -.2320 & \lfloor \underline{.04} \\
 & \underline{.04} & \underline{.186} & \underline{.1962} & \\
 & 4.64 & 4.906 & .0358 & \\
 & \underline{.04} & \underline{.187} & & \\
 & 4.68 & 5.093 & &
\end{array}
$$

$$\frac{.0358}{5.093} = .007$$

Hence the third root is -1.247. (Art. 488.) Instead of going through the operation as above, the last root may be found by subtracting the sum of the other two from 1. (Art. 441.)

2. As the term which precedes the missing term has the same sign as that which follows it, there must be two imaginary roots. By trial (Art. 474), the two real roots are found to lie between 2 and 3, and 3 and 4.

1	—4	±0	—3	+27	3.67963
	3	—3	—9	—36	
	—1	—3	—12	* — 9.0000	
	3	6	9	6.5376	
	2	3	* — 3.000	* — 2.4624	
	3	15	13.896	2.1402	
	5	* 18.00	10.896	* — .3222	
	3	5.16	17.208	.3008	
	* 8.0	23.16	* 28.104	* — .0214	
	.6	5.52	2.470	.0203	
	8.6	28.68	30.574	* — .0011	
	.6	5.88	2.522	.0010	
	9.2	* 34.56	* 33.096		
	.6	.73	.332		
	9.8	35.29	33.428		
	.6	.74	.333		
	* 10.40	36.03	* 33.8		
	.07	.74			
	10.47	* 36.77			
	.07	.09			
	10.54	36.9			

In the above operation the first decimal figure must be obtained by trial. (Art. 490.)

3. By Descartes' Rule (Art. 452), it is evident that there can be but one positive root; and this is found by trial (Art. 474), or by Sturm's Theorem, to be between 2 and 3.

1	±0	−2	−5	2.09455
	2	4	4	
	2	2	* −1.0000	
	2	8	.9493	
	4	* 10.000	* −.0507	
	2	.548	.0445	
	* 6.00	10.548	* −.0062	
	.09	.556	.0056	
	6.09	* 11.104	* −.0006	
	.09	.025	.0006	
	6.18	11.129		
	.09	.025		
	* 6.27	* 11.2		

That the above root is the only real one, may be proved by Sturm's Theorem, thus:

$$X = x^3 - 2x - 5; \qquad X_2 = 4x + 15;$$
$$X_1 = 3x^2 - 2; \qquad X_3 = -.$$

	X	X_1	X_2	X_3	
$x = -\infty$,	−	+	−	−,	2 variations.
$x = 2$,	−	+	+	−,	2 variations.
$x = 3$,	+	+	+	−,	1 variation.
$x = +\infty$,	+	+	+	−,	1 variation.

4. By Sturm's Theorem it is easily proved that the equation has but one real root, lying between 7 and 8, as follows:

$$X = x^3 + x^2 - 500; \qquad X_2 = x + 2250;$$
$$X_1 = 3x^2 + 2x; \qquad X_3 = -.$$

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$-$,	2 variations.
$x = 0$,	$-$	0	$+$	$-$,	2 variations.
$x = 7$,	$-$	$+$	$+$	$-$,	2 variations.
$x = 8$,	$+$	$+$	$+$	$-$,	1 variation.
$x = +\infty$,	$+$	$+$	$+$	$-$,	1 variation.

In order to find *one root*, it would only be necessary to substitute different positive numbers for x in X, until the sign is found to change (Art. 474), and Descartes' Rule proves that there can be only one *positive* root (Art. 452). The use of X_1, X_2, and X_3 proves that there can be no *negative* root, and that the other two roots are imaginary.

1	$+1$	± 0	-500	7.61728
	7	56	392	
	8	56	* -108.000	
	7	105	104.736	
	15	* 161.00	* -3.264	
	7	13.56	1.887	
	* 22.0	174.56	* -1.377	
	.6	13.92	1.324	
	22.6	* 188.48	* $-.053$	
	.6	.24	.038	
	23.2	188.72	* $-.015$	
	.6	.24	.015	
	* 23.8	* 188.96		
		.17		
		189.1		

5. By Sturm's Theorem it is easily shown that the equation has one negative root and two positive roots, both of the latter lying between 1 and 2.

$X = x^3 - 7x + 7$; $\quad X_2 = 2x - 3$;

$X_1 = 3x^2 - 7$; $\quad X_3 = +1$.

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$+$,	3 variations.
$x = 0$,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1$,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1.5$,	$-$	$-$	0	$+$,	1 variation.
$x = 2$,	$+$	$+$	$+$	$+$,	no variation.
$x = +\infty$,	$+$	$+$	$+$	$+$,	no variation.

By substituting 1.4, 1.3, 1.6, and 1.7 for x it is found that the second figures of the two positive roots are .3 and .6.

± 0	-7	$+7$	1.3569
1	1	-6	
1	-6	* 1.000	
1	2	$-.903$	
2	* -4.00	* .0970	
1	.99	$-.0866$	
* 3.0	-3.01	* .0104	
.3	1.08	$-.0090$	
3.3	* -1.930	* .0014	
.3	.198	$-.0014$	
3.6	-1.732		
.3	.200		
* 3.90	* -1.532		
.05	.024		
3.95	-1.508		
.05			
4.00			

1	±0	—7	+7	1.6920
	1	1	—6	
	1	—6	* 1.000	
	1	2	—1.104	
	2	* —4.00	* —.1040	
	1	2.16	.1008	
	* 3.0	—1.84	* —.0032	
	.6	2.52	.0032	
	3.6	* .68		
	.6	.44		
	4.2	1.12		
	.6	.45		
	* 4.80	* 1.57		
	.09	.01		
	4.89	1.58		
	.09			
	5.			

The negative root is the sum of the other two, with the sign changed (Art. 442), or —3.0489.

6. $X = x^3 - 17x^2 + 54x - 350$; $X_2 = 127x + 1116$;
$X_1 = 3x^2 - 34x + 54$; $X_3 = -$.

	X	X_1	X_2	X_3	
$x = -\infty$,	—	+	—	—,	2 variations.
$x = 0$,	—	+	+	—,	2 variations.
$x = +\infty$,	+	+	+	—,	1 variation.

Hence there is only one real root, and that is positive. By trial, the integral part of that root is found to be 14.

1	—17	+54	—350	14.95407
	14	—42	168	
	—3	12	* —182.000	
	14	154	170.379	
	11	* 166.00	* —11.621	
	14	23.31	10.741	
	* 25.0	189.31	* —.880	
	.9	24.12	.865	
	25.9	* 213.43	* —.015	
	.9	1.39	.015	
	26.8	214.82		
	.9	1.39		
	* 27.7	* 216.21		

7. By trial, the first figure of the root is found to be 9.

1	±0	—3	+75	—10000	9.8860027
	9	81	702	6993	
	9	78	777	* —3007.0000	
	9	162	2160	2677.5616	
	18	240	* 2937.000	* —329.4384	
	9	243	409.952	306.1662	
	27	* 483.00	3346.952	* —23.2722	
	9	29.44	434.016	23.2616	
	* 36.0	512.44	* 3780.968	* —.0106	
	.8	30.08	46.110	.0078	
	36.8	542.52	3827.078	* —.0028	
	.8	30.72	46.362	.0027	
	37.6	* 573.24	* 3873.44		
	.8	3.14	3.50		
	38.4	576.38	3876.94		
	.8	3.15	3.50		
	* 39.2	579.53	* 3880.		
	.08	3.			
	39.28	* 583.			

APPENDIX.

RECURRING OR RECIPROCAL EQUATIONS.

(Art. 497, pp. 383, 384.)

2. Given $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0$.

By Art. 494 it is evident that one root of the equation is -1; we therefore divide by $x + 1$.

Depressed equation, $x^4 - 12x^3 + 29x^2 - 12x + 1 = 0$

Dividing by x^2, $x^2 - 12x + 29 - \frac{12}{x} + \frac{1}{x^2} = 0$

Or, $\left(x^2 + \frac{1}{x^2}\right) - 12\left(x + \frac{1}{x}\right) + 29 = 0$

Substituting y for $x + \frac{1}{x}$, and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$,

$$y^2 - 12y + 27 = 0$$

Completing square, $y^2 - 12y + 36 = 9$

Whence, $y = 9$, or 3

Replacing value of y, $x + \frac{1}{x} = 9$, or 3

$$x^2 - 9x + 1 = 0, \quad \text{or } x^2 - 3x + 1 = 0$$

$$x^2 - 9x + \tfrac{81}{4} = \tfrac{77}{4}, \quad \text{or } x^2 - 3x + \tfrac{9}{4} = \tfrac{5}{4}$$

$$x = \frac{9 \pm \sqrt{77}}{2}, \text{ or } \quad x = \frac{3 \pm \sqrt{5}}{2}$$

3. Given $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

By Art. 494 one root is -1; we therefore divide by $x + 1$.

Depressed equation, $x^4+x^3-4x^2+x+1=0$

Dividing by x^2, $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$

Subst. y for $x+\frac{1}{x}$, $y^2+y-6=0$

Completing square, $y^2+y+\frac{1}{4}=\frac{25}{4}$

Whence, $y=2$, or -3

Replacing value of y, $x+\frac{1}{x}=2$, or -3

$x^2-2x+1=0$, or $x^2+3x+1=0$

$x-1=0$, or $x^2+3x+\frac{9}{4}=\frac{5}{4}$

$x=1$, or 1, or $x=\frac{-3\pm\sqrt{5}}{2}$

4. Given $x^6-x^5+x^4-x^2+x-1=0$.

By Art. 495, two of the roots are $+1$ and -1; we therefore divide by x^2-1.

Depressed equation, $x^4-x^3+2x^2-x+1=0$

Dividing by x^2, $\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)+2=0$

Subst. y for $x+\frac{1}{x}$, $y^2-y=0$

Whence, $y=0$, or 1

Replacing value of y, $x+\frac{1}{x}=0$, or 1

$x^2+1=0$, or $x^2-x+1=0$

$x^2=-1$, or $x^2-x+\frac{1}{4}=-\frac{3}{4}$

$x=\pm\sqrt{-1}$, or $x=\frac{1\pm\sqrt{-3}}{2}$

5. Given $x^3+px^2+px+1=0$.

By Art. 494, one root is -1; we therefore divide by $x+1$.

Depressed equation, $x^2+(p-1)x+1=0$

Completing square, $x^2+(p-1)x+\left(\frac{p-1}{2}\right)^2=\frac{p^2-2p-3}{4}$

Whence, $x=\frac{1}{2}\left(1-p\pm\sqrt{p^2-2p-3}\right)$

6. Given $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

Dividing by x^2, $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

Subst. y for $x + \frac{1}{x}$, and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$ (Art. 496),

$$6y^2 - 12 + 5y - 38 = 0$$

Or, $y^2 + \frac{5}{6}y - \frac{50}{6} = 0$

Completing square, $y^2 + \frac{5}{6}y + \frac{25}{144} = \frac{1225}{144}$

Whence, $y = \frac{-5 \pm 35}{12}$

Or, $y = \frac{5}{2}$, or $-\frac{10}{3}$

Replacing value of y, $x + \frac{1}{x} = \frac{5}{2}$, or $-\frac{10}{3}$

$x^2 - \frac{5}{2}x + 1 = 0$, or $x^2 + \frac{10}{3}x + 1 = 0$

$x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{9}{16}$, or $x^2 + \frac{10}{3}x + \frac{25}{9} = \frac{16}{9}$

$x = 2$, or $\frac{1}{2}$, or $x = -\frac{1}{3}$, or -3

7. Given $5x^5 - 51x^4 + 160x^3 - 160x^2 + 51x - 5 = 0.$

By Art. 494, one root is $+1$; we therefore divide by $x - 1$.

Depressed equation, $5x^4 - 46x^3 + 114x^2 - 46x + 5 = 0$

Dividing by x^2, $5\left(x^2 + \frac{1}{x^2}\right) - 46\left(x + \frac{1}{x}\right) + 114 = 0$

Subst. y for $1 + \frac{1}{x}$, $5y^2 - 46y + 104 = 0$

Or, $y^2 - \frac{46}{5}y + \frac{104}{5} = 0$

Completing square, $y^2 - \frac{46}{5}y + \frac{529}{25} = \frac{9}{25}$

Whence, $y = \frac{23 \pm 3}{5}$

Replacing value of y, $x + \frac{1}{x} = \frac{26}{5}$, or 4

$x^2 - \frac{26}{5}x + 1 = 0$, or $x^2 - 4x + 1 = 0$

$x^2 - \frac{26}{5}x + \frac{169}{25} = \frac{144}{25}$, or $x^2 - 4x + 4 = 3$

$x = \frac{13 \pm 12}{5}$, or $x = 2 \pm \sqrt{3}$

$x = 5$, or $\frac{1}{5}$

8. Given $x^4 + 5x^3 + 5x + 1 = 0$

Dividing by x^2, $\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) = 0$

Subst. y for $1 + \frac{1}{x}$, $y^2 + 5y - 2 = 0$

Completing square, $y^2 + 5y + \frac{25}{4} = \frac{33}{4}$

Whence, $y = \frac{1}{2}(-5 \pm \sqrt{33})$

Replacing value of y, $x + \frac{1}{x} = \frac{-5 \pm \sqrt{33}}{2}$

Whence, $x^2 + \left(\frac{5 \mp \sqrt{33}}{2}\right)x + 1 = 0$

Compl. square, $x^2 + \left(\frac{5 \mp \sqrt{33}}{2}\right)x + \left(\frac{5 \mp \sqrt{33}}{4}\right)^2 = \frac{42 \mp 10\sqrt{33}}{16}$

Evolving, $x + \frac{5 \mp \sqrt{33}}{4} = \pm\left(\frac{42 \mp 10\sqrt{33}}{16}\right)^{\frac{1}{2}}$

Or, $x = -\frac{5}{4} \pm \frac{1}{4}\sqrt{33} \pm \frac{1}{4}(42 \mp 10\sqrt{33})^{\frac{1}{2}}$

BINOMIAL EQUATIONS.

(Arts. **506**, **507**, pp. 386, 387.)

3. Let $x^3 = 1$, or $x^3 - 1 = 0$. Then, by Arts. 494, 501, one root of this equation is $+1$; we therefore divide $x^3 - 1$ by $x - 1$.

Depressed equation, $x^2 + x + 1 = 0$

Completing square, $x^2 + x + \frac{1}{4} = -\frac{3}{4}$

Whence, $x = \frac{1}{2}(-1 \pm \sqrt{-3})$

4. Given $x^3 + a^3 = 0$

Substituting $a\,y$ for x, $a^3 y^3 + a^3 = 0$

Dividing by a^3, $y^3 + 1 = 0$

By Arts. 494, 501, one root is -1; we therefore divide by $y + 1$, Depressed equation, $y^2 - y + 1 = 0$

Completing square, $y^2 - y + \frac{1}{4} = -\frac{3}{4}$

Whence, $y = \frac{1}{2}(1 \pm \sqrt{-3})$

Multiplying by a, $ay = x = \frac{a}{2}(1 \pm \sqrt{-3})$

It is also evident that if one value of y is -1, one value of x must be $-a$.

5. Let $x^4 = 1$, or $x^4 - 1 = 0$. Then, by Arts. 495, 501, the only real roots are $+1$ and -1; we therefore divide by $x^2 - 1$.

Depressed equation, $x^2 + 1 = 0$

Whence, $x = \pm\sqrt{-1}$

6. By Arts. 494, 501, the only real root is -1; we therefore divide by $x + 1$.

Depressed equation, $x^4 - x^3 + x^2 - x + 1 = 0$

Dividing by x^2, $\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 1 = 0$

Substituting y for $x + \frac{1}{x}$, $y^2 - y - 1 = 0$

Completing square, $y^2 - y + \frac{1}{4} = \frac{5}{4}$

Whence, $y = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

Replacing value of y, $x + \frac{1}{x} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

Or, $x^2 - \frac{1}{2}(1 \pm \sqrt{5})\,x + 1 = 0$

Compl. square, $x^2 - \frac{1}{2}(1 \pm \sqrt{5})\,x + \frac{1}{16}(1 \pm \sqrt{5})^2 = \frac{-10 \pm 2\sqrt{5}}{16}$

Whence, $x = \frac{1}{4}\left(1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}}\right)$

The above example may also be solved like Example 2.

7. The factors of $x^6 - a^6$ are $x^3 + a^3$ and $x^3 - a^3$. When $x^3 + a^3 = 0$, $x = -a$, and $\frac{a}{2}(1 \pm \sqrt{-3})$, by Example 4. When $x^3 - a^3 = 0$, the substitution of $a\,y$ for x gives $y^3 - 1 = 0$,

in which $y=1$, and $-\frac{1}{2}(1\pm\sqrt{-3})$, by Example 3, and multiplying by a, $x=a$, and $-\frac{a}{2}(1\pm\sqrt{-3})$.

Hence $x=\pm a$, and $\pm\frac{a}{2}(1\pm\sqrt{-3})$.

The above equation may also be solved by means of Arts. 495, 500, 501, without separating it into two factors of the third degree.

8. Substituting $2y$ for $x^{\frac{1}{2}}$, we have $8y^3-8=0$, or $y^3-1=0$, whose roots are 1, and $\frac{1}{2}(-1\pm\sqrt{-3})$, by Example 3. Multiplying by 2, and replacing the value of y,

$$2y=x^{\frac{1}{2}}=2, \text{ and } -1\pm\sqrt{-3}$$

Squaring, $$x=4, \text{ and } -2\mp2\sqrt{-3}$$

9. Substituting $3y$ for $x^{\frac{1}{3}}$, we have $81y^4-81=0$, or $y^4-1=0$, whose roots are ±1, and $\pm\sqrt{-1}$ by Example 5. Multiplying by 3, and replacing the value of y,

$$3y=x^{\frac{1}{3}}=\pm3, \text{ and } \pm3\sqrt{-1}$$

Cubing (Art. 256), $$x=\pm27, \text{ and } \mp27\sqrt{-1}$$

As the sign ∓ in the last two examples is independent of all others in the same expression, it has the same force as ±.

CARDAN'S RULE FOR CUBICS.

(ART. **513**, pp. 390, 391.)

3. Here $p=6$ and $q=-2$. Substituting in formula (B), Art. 508,

$$x=(1+\sqrt{1+8})^{\frac{1}{3}}+(1-\sqrt{1+8})^{\frac{1}{3}}$$

Reducing, $$x=\sqrt[3]{4}+\sqrt[3]{-2}$$

Or, $$x=\sqrt[3]{4}-\sqrt[3]{2}$$

4. Given $x^3 - 6x + 9 = 0$

Substituting $z + \frac{2}{z}$ for x, $z^3 + \frac{8}{z^3} + 9 = 0$

Or, $z^6 + 9z^3 + 8 = 0$

Compl. square, $z^6 + 9z^3 + \frac{81}{4} = \frac{49}{4}$

Whence, $z^3 + \frac{9}{2} = \pm\frac{7}{2}$

Then, $z^3 = -1$, or -8

And $z = -1$, or -2

Substituting, $x = z + \frac{2}{z} = -1 - 2$, or $-2 - 1$

Or, $x = -3$

Dividing $x^3 - 6x + 9$ by $x + 3$, we obtain

Depressed equation, $x^2 - 3x + 3 = 0$

Completing square, $x^2 - 3x + \frac{9}{4} = -\frac{3}{4}$

Whence, $x = \dfrac{3 \pm \sqrt{-3}}{2}$

5. Substituting $y + 2$ for x, in order to remove the second term (Art. 466), we obtain

$$y^3 + 45y - 98 = 0.$$

The substitution may best be effected by synthetic division, as is shown in Example 9.

Substituting 45 for p, and -98 for q, in formula (B),

$$y = (49 + \sqrt{2401 + 3375})^{\frac{1}{3}} + (49 - \sqrt{2401 + 3375})^{\frac{1}{3}}$$

Or, $y = \sqrt[3]{49 + 76} + \sqrt[3]{49 - 76} = 5 - 3 = 2$

Hence, $x = y + 2 = 2 + 2 = 4$

The same result may be obtained by substituting $z - \frac{15}{z}$ for y, and going through the operation according to the Rule, as in the 4th example.

Dividing $x^3 - 6x^2 + 57x - 196$ by $x - 4$, we obtain

Depressed equation, $x^2-2x+49=0$

Completing square, $x^2-2x+1=-48$

Whence, $x=1\pm 4\sqrt{-3}$

6. Substituting $y+\frac{4}{3}$ for x, in order to remove the second term (Art. 466), we obtain

$$y^3-\tfrac{25}{3}y+\tfrac{250}{27}=0.$$

Substituting $-\frac{25}{3}$ for p, and $\frac{250}{27}$ for q, in formula (B),

$$y=\left(-\tfrac{125}{27}+\sqrt{\tfrac{15625}{729}-\tfrac{15625}{729}}\right)^{\frac{1}{3}}+\left(-\tfrac{125}{27}-\sqrt{\tfrac{15625}{729}-\tfrac{15625}{729}}\right)^{\frac{1}{3}}$$

Or, $y=\sqrt[3]{-\tfrac{125}{27}}+\sqrt[3]{-\tfrac{125}{27}}=-\tfrac{5}{3}-\tfrac{5}{3}=-\tfrac{10}{3}$

Hence, $x=y+\tfrac{4}{3}=-\tfrac{10}{3}+\tfrac{4}{3}=-\tfrac{6}{3}=-2$

Dividing $x^3-4x^2-3x+18$ by $x+2$, we obtain

Depressed equation, $x^2-6x+9=0$

Whence, $x=3$, or 3

7. Substituting $y+8$ for x,

$$y^3-216y-1241=0$$

Substituting -216 for p, and -1241 for q, in formula (B),

$$y=\left(\tfrac{1241}{2}+\sqrt{\tfrac{1540081}{4}-373248}\right)^{\frac{1}{3}}+\left(\tfrac{1241}{2}-\sqrt{\tfrac{1540081}{4}-373248}\right)^{\frac{1}{3}}$$

Or, $=\left(\tfrac{1241}{2}+\tfrac{217}{2}\right)^{\frac{1}{3}}+\left(\tfrac{1241}{2}-\tfrac{217}{2}\right)^{\frac{1}{3}}=\sqrt[3]{729}+\sqrt[3]{512}$

$=9+8=17$

Hence, $x=y+8=17+8=25$

Dividing $x^3-24x^2-24x-25$ by $x-25$, we obtain

Depressed equation, $x^2+x+1=0$

Completing square, $x^2+x+\frac{1}{4}=-\frac{3}{4}$

Whence, $x=\dfrac{-1\pm\sqrt{-3}}{2}$

8. Substituting $y-3$ for x,

$$y^3-48y+128=0$$

Substituting -48 for p, and 128 for q, in formula (B),

$$y = (-64 + \sqrt{4096 - 4096})^{\frac{1}{3}} + (-64 - \sqrt{4096 - 4096})^{\frac{1}{3}}$$

Or, $y = \sqrt[3]{-64} + \sqrt[3]{-64} = -4 - 4 = -8$

Hence, $x = y - 3 = -8 - 3 = -11.$

Dividing $x^3 + 9x^2 - 21x + 11$ by $x + 11$,

Depressed equation, $x^2 - 2x + 1 = 0$

Whence, $x = 1$, or 1

9. Substituting $y + \frac{2}{3}$ for x,

$$\begin{array}{ccccl}
1 & -2 & +2 & -1 & \underline{|+\frac{2}{3}} \\
 & \frac{2}{3} & -\frac{8}{9} & +\frac{20}{27} & \\ \cline{2-4}
 & -\frac{4}{3} & +\frac{10}{9}, & -\frac{7}{27} & \\
 & +\frac{2}{3} & -\frac{4}{9} & & \\ \cline{2-3}
 & -\frac{2}{3}, & +\frac{2}{3} & & \\
 & +\frac{2}{3} & & & \\ \cline{2-2}
 & 0 & & &
\end{array}$$

Hence $y^3 + \frac{2}{3}y - \frac{7}{27} = 0.$

Substituting $\frac{2}{3}$ for p, and $-\frac{7}{27}$ for q, in formula (B),

$$y = \left(\tfrac{7}{54} + \sqrt{\tfrac{49}{2916} + \tfrac{8}{729}}\right)^{\frac{1}{3}} + \left(\tfrac{7}{54} - \sqrt{\tfrac{49}{2916} + \tfrac{8}{729}}\right)^{\frac{1}{3}}$$

Or, $$y = \left(\tfrac{7}{54} + \sqrt{\tfrac{729}{26244}}\right)^{\frac{1}{3}} + \left(\tfrac{7}{54} - \sqrt{\tfrac{729}{26244}}\right)^{\frac{1}{3}}$$

$$= \left(\tfrac{7}{54} + \tfrac{27}{162}\right)^{\frac{1}{3}} + \left(\tfrac{7}{54} - \tfrac{27}{162}\right)^{\frac{1}{3}} = \sqrt[3]{\tfrac{8}{27}} + \sqrt[3]{-\tfrac{1}{27}}$$

$$= \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3}$$

Hence, $x = y + \frac{2}{3} = \frac{1}{3} + \frac{2}{3} = 1$

Dividing $x^3 - 2x^2 + 2x - 1$ by $x - 1$, we obtain

Depressed equation, $x^2 - x + 1 = 0$

Completing square, $x^2 - x + \frac{1}{4} = -\frac{3}{4}$

Whence, $x = \dfrac{1 \pm \sqrt{-3}}{2}$

The above equation may also be solved as a recurring one.

10. Substituting 6 for p, and -20 for q, in formula (B),

$$x = (10 + \sqrt{100+8})^{\frac{1}{3}} + (10 - \sqrt{100+8})^{\frac{1}{3}}$$

Or, $x = (10 + 6\sqrt{3})^{\frac{1}{3}} + (10 - 6\sqrt{3})^{\frac{1}{3}} = 1 + \sqrt{3} + 1 - \sqrt{3} = 2$

Dividing $x^3 + 6x - 20$ by $x - 2$, we obtain

Depressed equation, $x^2 + 2x + 10 = 0$
Completing square, $x^2 + 2x + 1 = -9$
Whence, $x = -1 \pm \sqrt{-9} = -1 \pm 3\sqrt{-1}$

APPROXIMATION BY DOUBLE POSITION.

(Art. **515**, pp. 392, 393.)

2. When 3 and 4 are substituted for x in the equation we have

$$\begin{array}{rrcr}
x^3 = & +27, & \text{and} & +64 \\
-2x = & -\ \ 6, & \text{and} & -\ \ 8 \\
 & \underline{-50} & & \underline{-50} \\
\text{Errors} = & -29, & \text{and} & +6
\end{array}$$

It is evident that 4 is nearer the true root than 3; hence $a = 4$, $b = 3$, $A = 6$, and $B = -29$. Substituting these values in the formula,

$$x = 4 + \frac{6\,(3-4)}{6+29} = 4 + \frac{-6}{35} = 4 - (.2-) = 3.8+$$

When 3.8 and 3.9 are substituted for x, we have

$$\begin{array}{rrcr}
x^3 = & +54.872, & \text{and} & +59.319 \\
-2x = & -\ \ 7.6, & \text{and} & -\ \ 7.8 \\
 & \underline{-50.0} & & \underline{-50.0} \\
\text{Errors} = & -\ \ 2.728, & \text{and} & +\ \ 1.519
\end{array}$$

As 3.9 is nearer the true root than 3.8, $a = 3.9$, $b = 3.8$, $A = 1.519$, and $B = -2.728$. Substituting these values in the formula,

$$x = 3.9 + \frac{1.52\,(3.8 - 3.9)}{1.52 + 2.73} = 3.9 + \frac{-.152}{4.25}$$

$$= 3.9 - (.036\,-) = 3.864 +$$

When 3.864 and 3.865 are substituted for x, we have

$$\begin{array}{rrr}
x^3 = & 57.69144, \text{ and} & 57.73624 \\
-2x = & -7.72800, \text{ and} & -7.73000 \\
 & \underline{-50.00000} & \underline{-50.00000} \\
\text{Errors} = & -.03656, \text{ and} & +.00624
\end{array}$$

Here $a = 3.865$, $b = 3.864$, $A = .00624$, and $B = -.03656$. Substituting in the formula,

$$x = 3.865 + \frac{.00624\,(3.864 - 3.865)}{.00624 + .03656} = 3.865 + \frac{-.00000624}{.0428}$$

$$= 3.865 - (.000146\,-) = 3.864854 +$$

3. When 4 and 5 are substituted for x in the equation, we have

$$\begin{array}{rrr}
x^3 = & 64, \text{ and} & 125 \\
10x^2 = & 160, \text{ and} & 250 \\
5x = & 20, \text{ and} & 25 \\
 & \underline{-260,} & \underline{-260} \\
\text{Errors} = & -16, \text{ and} & +140
\end{array}$$

The difference of the errors is $140 + 16$, or 156.

Then, $156 : 1 :: 16 : .1$, the correction.

Hence, $x = 4 + (.1 +) = 4.1 +$

When 4.1 and 4.2 are substituted for x, we have

$$\begin{array}{rrr}
x^3 = & 68.92, \text{ and} & 74.09 \\
10x^2 = & 168.10, \text{ and} & 176.40 \\
5x = & 20.50, \text{ and} & 21.00 \\
 & \underline{-260.00,} & \underline{-260.00} \\
\text{Errors} = & -2.48, \text{ and} & +11.49
\end{array}$$

The difference of the errors is $11.49 + 2.48$, or 13.97.

Then, $13.97 : .1 :: 2.48 : .017 +$

Hence, $x = 4.1 + (.017 +) = 4.117 +$

When 4.117 and 4.118 are substituted for x, we have

$$\begin{array}{rrr}
x^3 = & 69.78187, \text{ and} & 69.83273 \\
10x^2 = & 169.49689, \text{ and} & 169.57924 \\
5x = & 20.58500, \text{ and} & 20.59000 \\
& -260. & -260. \\
\hline
\text{Errors} = & -.13624, \text{ and} & .00197
\end{array}$$

The difference of the errors is $.13624 + .00197$, or $.13821$.

$$\text{Then, } .13821 : .001 :: .00197 : .000014$$

$$\text{Hence, } x = 4.118 - .000014 = 4.117986$$

4. When 2 and 3 are substituted for x in the equation, we have

$$\begin{array}{rrr}
x^3 = & 8.0, \text{ and} & 27.0 \\
8x^2 = & 32.0, \text{ and} & 72.0 \\
6x = & 12.0, \text{ and} & 18.0 \\
& -75.9 & -75.9 \\
\hline
\text{Errors} = & -23.9, \text{ and} & +41.1
\end{array}$$

The difference of the errors is $23.9 + 41.1$, or 65.

$$\text{Then, } 65 : 1 :: 23.9 : .4-$$

$$\text{Hence, } x = 2 + (.4-) = 2.4-$$

When 2.3 and 2.4 are substituted for x, we have

$$\begin{array}{rrr}
x^3 = & 12.167, \text{ and} & 13.824 \\
8x^2 = & 42.320, \text{ and} & 46.080 \\
6x = & 13.800, \text{ and} & 14.400 \\
& -75.900 & -75.900 \\
\hline
\text{Errors} = & -7.613, \text{ and} & -1.596
\end{array}$$

The difference of the errors is $7.613 - 1.596 = 6.017$.

$$\text{Then, } 6.017 : .1 :: 1.596 : .02+$$

$$\text{Hence, } x = 2.4 + (.02+) = 2.42+$$

When 2.42 and 2.43 are substituted for x, we have

$$\begin{array}{rll}
x^3 = & 14.1725, \text{ and} & 14.3489 \\
8x^2 = & 46.8512, \text{ and} & 47.2392 \\
6x = & 14.5200, \text{ and} & 14.5800 \\
 & \underline{-75.9} & \underline{-75.9} \\
\text{Errors} = & -.3563, \text{ and} & +.2681
\end{array}$$

The difference of the errors is $.3563 + .2681$, or $.6244$.

$$\text{Then, } .6244 : .01 :: .2681 : .0043$$

$$\text{Hence, } x = 2.43 - .0043 = 2.4257$$

5. When .6 and .7 are substituted for x in the equation, we have

$$\begin{array}{rll}
x^3 = & .22, \text{ and} & .34 \\
\frac{11}{16}x = & .41, \text{ and} & .48 \\
 & \underline{-.75} & \underline{-.75} \\
\text{Errors} = & -.12, \text{ and} & +.07
\end{array}$$

The difference of the errors is .19.

$$\text{Then, } .19 :: .1 :: .07 : .04-$$

$$\text{Hence, } x = .7 - (.04-) = .66+$$

When .66 and .67 are substituted for x, we have

$$\begin{array}{rll}
x^3 = & .287496, \text{ and} & .300763 \\
\frac{11}{16}x = & .453750, \text{ and} & .460625 \\
 & \underline{-.750000} & \underline{-.750000} \\
\text{Errors} = & -.008754, \text{ and} & +.011388
\end{array}$$

The difference of the errors is .020142.

$$\text{Then, } .020142 : .01 :: .008754 : .0043+$$

$$\text{Hence, } x = .66 + (.0043+) = .6643+$$

When .664 and .665 are substituted for x, we have

$$\begin{array}{rll}
x^3 = & .292755, \text{ and} & .294080 \\
\frac{11}{16}x = & .456500, \text{ and} & .457188 \\
 & \underline{-.750000} & \underline{-.750000} \\
\text{Errors} = & -.000745, \text{ and} & +.001268
\end{array}$$

The difference of the errors is .002013.

$$\text{Then, } .002013 : .001 :: .000745 : .00037$$
$$\text{Hence, } x = .664 + .00037 = .66437$$

6. When 10 and 11 are substituted for x in the equation, we have

$$\begin{array}{rrr}
x^4 = & 10000, \text{ and} & 14641 \\
-3x^2 = & -300, \text{ and} & -363 \\
-75x = & -750, \text{ and} & -825 \\
 & -10000 & -10000 \\
\hline
\text{Errors} = & -1050, \text{ and} & +3453
\end{array}$$

The difference of the errors is 4503.

$$\text{Then, } 4503 : 1 :: 1050 : .2+$$
$$\text{Hence, } x = 10 + (.2+) = 10.2+$$

When 10.2 and 10.3 are substituted for x, we have

$$\begin{array}{rrr}
x^4 = & 10824.32, \text{ and} & 11255.09 \\
-3x^2 = & -312.12, \text{ and} & -318.27 \\
-75x = & -765.00, \text{ and} & -772.50 \\
 & -10000.00 & -10000.00 \\
\hline
\text{Errors} = & -252.80, \text{ and} & +164.32
\end{array}$$

The difference of the errors is 417.12.

$$\text{Then, } 417.12 : .1 :: 164.32 : .039+$$
$$\text{Hence, } x = 10.3 - (.039+) = 10.261-$$

By substituting 10.26 and 10.27, the next figures of the root may be obtained.

7. When 8 and 9 are substituted for x in the equation, we have

$$\begin{array}{rrlr}
x^5 = & 32768, & \text{and} & 59049 \\
2x^4 = & 8192, & \text{and} & 13122 \\
3x^3 = & 1536, & \text{and} & 2187 \\
4x^2 = & 256, & \text{and} & 324 \\
5x = & 40, & \text{and} & 45 \\
 & -54321 & & -54321 \\
\hline
\text{Errors} = & -11529, & \text{and} & +20406
\end{array}$$

The difference of the errors is 31935.

$$\text{Then, } 31935 : 1 :: 11529 : .4-$$

Hence $x = 8.4$, nearly. After substituting 8.4, it is found too small, instead of too large; hence we use 8.5 for the other value. When 8.4 and 8.5 are substituted for x, we have

$$\begin{array}{rrlr}
x^5 = & 41821.194, & \text{and} & 44370.531 \\
2x^4 = & 9957.427, & \text{and} & 10440.125 \\
3x^3 = & 1778.112, & \text{and} & 1842.375 \\
4x^2 = & 282.240, & \text{and} & 289.000 \\
5x = & 42.000, & \text{and} & 42.500 \\
 & -54321.000 & & -54321.000 \\
\hline
 & -\ 440.027, & \text{and} & +\ 2663.531
\end{array}$$

The difference of the errors is 3103.558.

$$\text{Then, } 3103.558 : .1 :: 440.027 : .014+$$

$$\text{Hence, } x = 8.4 + (.014+) = 8.414+$$

By substituting 8.41 and 8.42, or 8.414 and 8.415, for x, the other figures of the root may be obtained.

NEWTON'S METHOD OF APPROXIMATION.

(Art. **516**, p. 394.)

2. When 1 and 2 are substituted for x in the equation, the results are -1 and 3, respectively; hence a root lies between 1 and 2. Substituting $1+y$ for x, there results

$$1 - 3y^2 - y^3 = 0,$$

whence, approximately, $y^2 = \frac{1}{3}$, $y = \sqrt{\frac{1}{3}} = .5$, and $x = 1 + .5 = 1.5$.

Substituting $1.5 + z$ for x, there results

$$.125 - 3.75\,z - \ldots\ldots\ldots = 0,$$

whence, approximately, $z = \frac{.125}{3.75} = .03$ and $x = 1.5 + .03 = 1.53$.

Substituting $1.53 + w$ for x, there results

$$.008423 - 4.0227\,w - \ldots\ldots = 0,$$

whence, approximately, $w = \frac{.008423}{4.0227} = .00209$, and

$$x = 1.53 + .00209 = 1.53209.$$

3. When 1 and 2 are substituted for x in the equation, the results are -1 and 24, respectively; hence a root lies between 1 and 2, and nearer to 1. Substituting $1 + y$ for x, there results

$$1 - 36y + 12y^2 - y^3 = 0,$$

whence, approximately, $y = \frac{1}{36} = .03$, and $x = 1 + .03 = 1.03$.

Substituting $1.03 + z$ for x, there results

$$.069227 + 35.2827\,z - \ldots\ldots = 0,$$

whence, approximately, $z = -\frac{.069227}{35.2827} = -.00196$ and

$$x = 1.03 - .00196 = 1.02804.$$

THE END.

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